Resonance frequency dependence of output power in inductively coupled circuits

Jorge Luis González Rios, Héctor Trujillo Alvarado
Centro de Investigaciones en Microelectrónica, Antigua Carretera de Vento, km 8
10800, Boyeros, Ciudad de la Habana, Cuba
{jorgeluis.gr, hector.trujillo}@electrica.cujae.edu.cu

Abstract

Resonance frequency dependence of output power in inductively coupled circuits is studied through Circuits Theory. The maximum output power theoretical value is obtained, as well as the optimal resonance frequency as a function of coils separation and load resistance, by using an ideal model for coupled coils. Theoretical analysis of a real system is performed, using the transmission system open-circuit impedance matrix. This method could lead to a faster way for tuning up real systems in order to achieve the maximum output power. Experimental results for output power prove the theoretical behavior of optimal resonance frequency.

Keywords: inductive link, coupled resonators, optimal resonance frequency, maximum output power

Introduction

Wireless powering is a requirement of many electronic devices and systems nowadays, i.e. mobile applications or wireless sensors. Using inductively coupled resonant elements is an efficient energy transfer method, based on non-radiative field interactions [1, 2]. The analysis of this kind of systems can be carried out through Circuits Theory, since system dimensions should be much smaller than the wave length to guarantee the non-radiative scheme. This will result in an easier way for optimizing output power, by leaving the physical analysis just for modeling the transmission system (transmitting and receiving coupled coils). In this paper we study how the optimal resonance frequency (that is, resonance frequency which leads to maximum output power) is affected by the distance between transmitting and receiving coils and by the value of load resistance. A method for predicting this optimal resonance frequency is also presented and used for the theoretical study of power transfer behavior.

Analysis using a linear transformer model

As a first approach, transmission system has been modeled as a linear transformer as shown in figure 1, by means of two coupled inductors (self inductances L_1, L_2 and mutual inductance M) and resistors R_1 and R_2 which take losses into account. They are connected in series with resonant capacitors (C_1 and C_2), source (represented by its Thévenin equivalent circuit, V_S and R_S) and load (R_L).

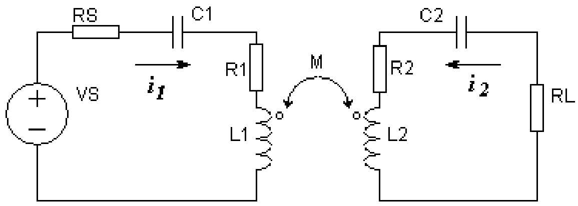

Fig. 1: Equivalent circuit using a linear transformer model.

The output power (P_L) can be obtained by:

$$P_L = \frac{1}{2}|I_2|^2 R_L \quad (1)$$

From this circuit the following set of equations can be written, using fasor notation:

$$\begin{gathered}\left[R_S + R_1 + j\left(\omega L_1 - \frac{1}{\omega C_1}\right)\right] I_1 + j\omega M I_2 = V_S \\ j\omega M I_1 + \left[R_L + R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] I_2 = 0\end{gathered} \quad (2)$$

Solving for I_2, and considering values for C_1 and C_2 that give the same resonance frequency in both coils $(\omega_0 = 1/\sqrt{L_1C_1} = 1/\sqrt{L_2C_2})$, the following expression for output power can be written:

$$P_L(\omega_0) = \frac{R_L}{2}\left[\frac{\omega_0 M |V_S|}{(R_1 + R_S)(R_2 + R_L) + (\omega_0 M)^2}\right]^2 \quad (3)$$

Optimizing the above expression is not an easy task, because ac resistance is frequency dependent. Therefore, let us consider a particular case, the lossless coils, making $R_1 << R_S$ and $R_2 << R_L$. By taking derivative of (3) equal to zero, the optimal resonance frequency can be found as:

$$\omega_{0opt} = \frac{1}{M}\sqrt{R_S R_L} \quad (4)$$

With this value of ω_0 the maximum output power is achieved:

$$P_{Lmáx} = \frac{|V_S|^2}{8R_S} \quad (5)$$

We have found that, in the ideal case, **maximum output power is independent of load and coupling**, so, **we can achieve the same power at any distance by selecting the proper resonance frequency**. By the other hand, according to (4), the optimal resonance frequency will be higher as values of distance and load are increased.

PSpice simulations were carried out in order to verify these results. We set $L_1 = L_2 =$ 70.5 µH, $V_S =$ 10 V and $R_S =$ 50 Ω, with a mutual inductance of 6.36 and 0.51 µH. These are the theoretical values of the real system used for the experimental investigation. We performed an AC sweep for both values of coupling, with a parametric sweep of capacitors, for achieving different resonance frequencies. Figure 2 shows power frequency response for a 22Ω load. We found $P_{Lmáx}$ = 250 mW for f_0 = 0.87 and 10.37 MHz, as expected in (4) and (5). This analysis was repeated for 56 and 100Ω loads, also with excellent agreement with those expresions.

In these simulations we find that, for resonance frequencies higher than optimal, the frequency response has two peaks around resonance. This result is not significant for our study because peaks are below maximum power too.

This ideal case, however, won't be valid at higher frequencies. At some point ac losses could not be neglected, stray capacitances' effects will show up (modeled in figure 3) as we come closer to coils' self-resonance frequency, $SRF \approx 1/\left(2\pi\sqrt{LC_p}\right)$, and radiation effects could also appear. Each of these effects will change the circuit model in a different way (radiation effects will actually prevent the use of lumped parameters model), so, a different and more complex system of equations should be solved in order to find optimal resonance frequency.

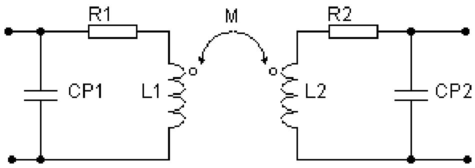

Fig. 3: Model including stray capacitances.

Analysis using open-circuit impedance parameters

Analysis based on open-circuit impedance parameters (*z* parameters) was chosen to give an unique solution to optimization problem, no matter how the transmission system has been modeled or characterized. In figure 4, coupled coils have been replaced by a two ports network, having the voltage-current relationship written in equation (6).

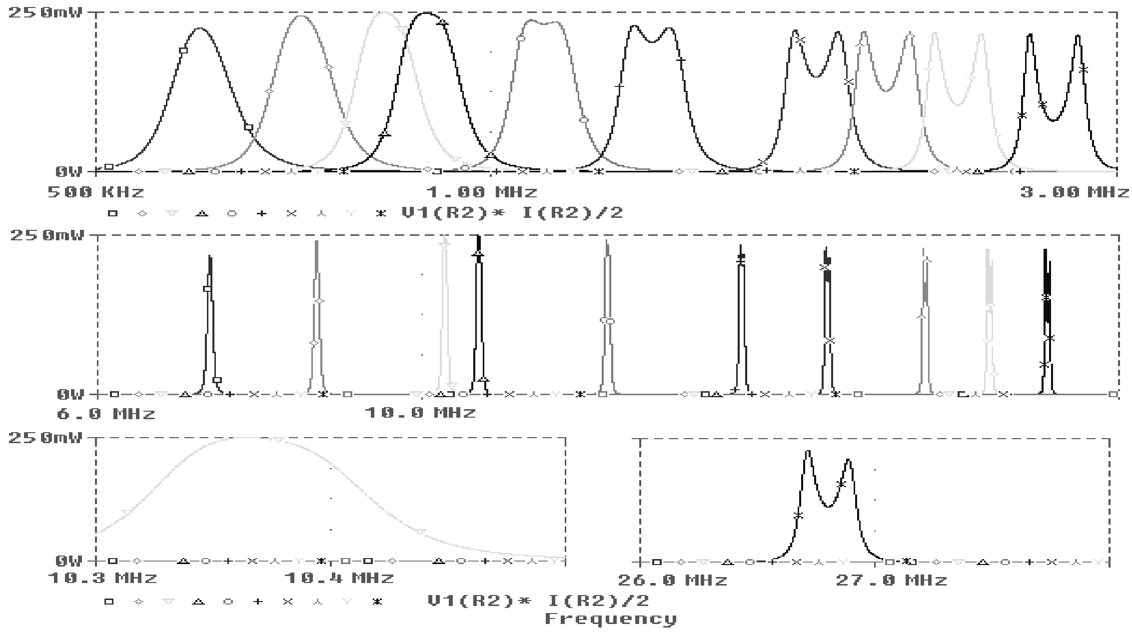

Fig. 2: PSpice simulation for ideal coils. Output power with R_L = 22 Ω, M = 6.36 µH (first row) y M = 0.51 µH (second and third rows).

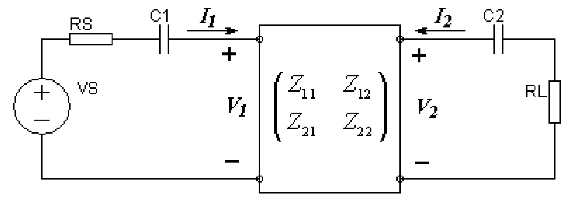

Fig. 4: Transmission system replaced by its z parameters.

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad (6)$$

The following system of equations can be achieved, where Z_s and Z_L have been defined as $R_s - j\left(1/\omega C_1\right)$ and $R_L - j\left(1/\omega C_2\right)$, respectively.

$$\begin{pmatrix} Z_{11}+Z_S & Z_{12} \\ Z_{21} & Z_{22}+Z_L \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_S \\ 0 \end{pmatrix} \quad (7)$$

Solving for output current:

$$I_2 = -\frac{Z_{21}V_S}{\left(Z_{11}+Z_S\right)\left(Z_{22}+Z_L\right) - Z_{12}Z_{21}} \quad (8)$$

Output power can be finally found by substituting (8) into (1).

Now, having coils' *z* parameters for several frequency values, a numerical method may be used for finding optimal resonance frequency. *Z* parameters can be obtained not only from circuit models, such as the linear transformer or even including stray capacitors, but also from computational electromagnetic field simulations or experimental measurements. Therefore this is a general method, fully compatible with diverse characterization techniques. A set of functions were programmed in MATLAB to calculate output power for different values of frequency, load and distances, and to find maximum output power and optimal resonance frequency as well. These functions allow a direct solution for resonance frequency, by setting capacitors reactance as $X_1(\omega) = -\mathrm{Im}\left[z_{11}(\omega)\right]$ and $X_2(\omega) = -\mathrm{Im}\left[z_{22}(\omega)\right]$, resulting in a faster method.

This method was verified by solving for the ideal coils' values used before, achieving the same results but in a shorter time.

Theoretical analysis

Two identical helical coils were built and analyzed. They were coaxially arranged, as shown in figure 5. We used coils radius *r* = 104 mm, copper diameter *a* = 2.65 mm, interwinding center to center separation *s* = 4.66 mm and turn number *N* = 17. Distance between coils (*d*) was set to 1, 2, 3 and 4 times coils radius (10.4, 20.8, 31.2 and 41.6 cm).

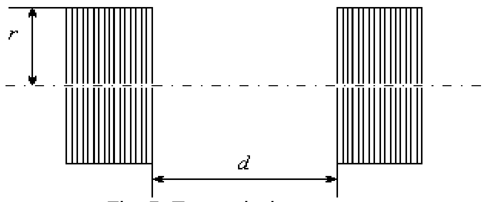

Fig. 5: Transmission system.

Expresions for sigle circular loops self (9) [3] and mutual (10) [4] inductances were used:

$$L_{single} = \mu_o r\left[\ln\left(\frac{8r}{a}\right) - 2\right] \quad (9)$$

$$M_{single} = \mu_0\sqrt{r_1 r_2}\left[\left(\frac{2}{k} - k\right)K(k) - \frac{2}{k}E(k)\right] \quad (10)$$

Where $\mu_0 = 4\pi \cdot 10^{-7}$ H/m. $K(k)$ and $E(k)$ are the Complete Elliptic Integrals:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2 sen^2\phi}} \quad (11)$$

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2 sen^2\phi}\, d\phi \quad (12)$$

With $k^2 = 4r_1 r_2 \Big/ \left(d^2 + \left(r_1 + r_2\right)^2\right)$.

From (9) and (10), expressions for self and mutual inductances for coaxial helical coils can be derived:

$$L = N \cdot L_{single} + 2[\left(N-1\right)\left(M_{12}\right) + \left(N-2\right)\left(M_{13}\right) + \ldots + \left(N-i+1\right)\left(M_{1i}\right) + \ldots + M_{1N}] \quad (13)$$

$$M = M_{(1.1)(2.1)} + M_{(1.1)(2.2)} + \ldots + M_{(1.1)(2.N_2)} + \ldots + M_{(1.2)(2.2)} + \ldots + M_{(1.N_1)(2.N_2)} \quad (14)$$

Where M_{ij} and $M_{(1.i)(2.j)}$ are mutual inductances between turns in the same and different coils respectively.

There are not analytical expressions for ac losses, due to proximity effects [5], so Maxwell 2D SV software, based on Finite Elements Method, was used for obtaining this parameter. This software was also used for turn-to-turn stray capacitances calculation [6]. Maxwell simulations also show that frequency dependence of inductances could be neglected.

With theoretical model the z matrix of coupled coils was created (using Matlab functions programmed by the authors for this purpose) for the four distances previously analyzed. We used the same values for generator and loads. Two sets of simulations were performed. The first didn't take into account stray capacitances, in order to study the effect of frequency dependent losses. For the

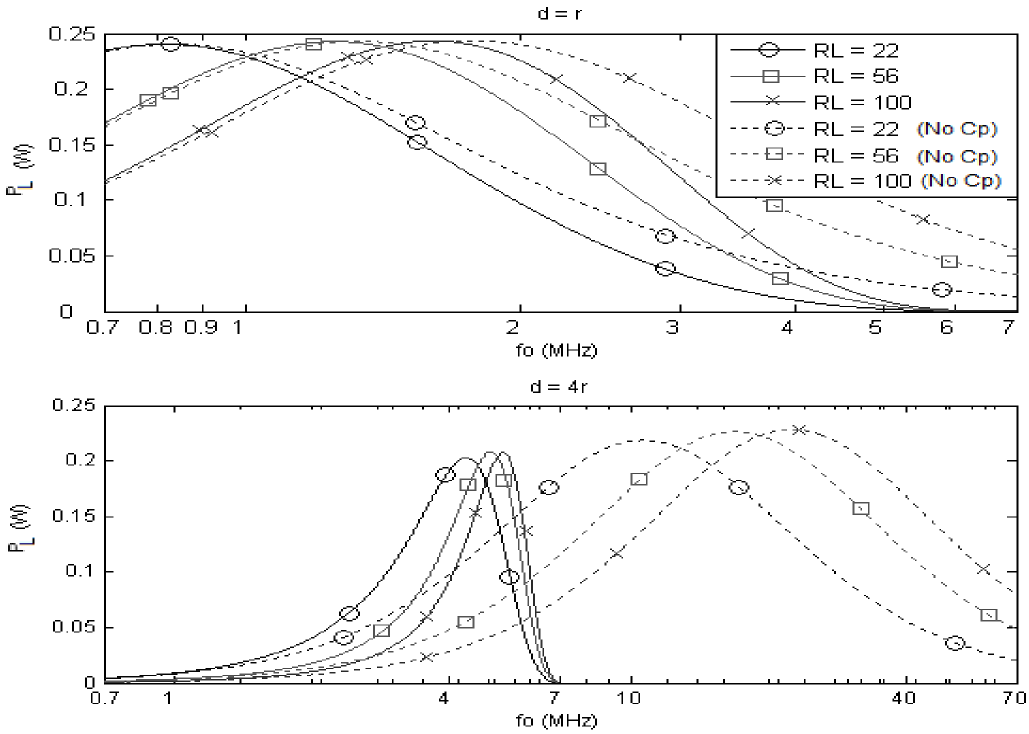

Fig. 6: Output power for theoretical model.

second simulation, z matrix was built from the complete model for real inductors. Figure 6 shows the results for $d = r$ and $4r$.

Optimal resonance frequency behavior was similar to that of the ideal case: as higher the values of distances between coils, the higher the resonance frequency required. However, the output power decreases as the distance is increased. This is a logical result for the real system, as the ac losses will be higher when the circuit is tuned up to a higher frequency. By the other hand, when stray capacitances are taken into account a limit for working frequency is introduced, that is, output power tends to zero when the resonance frequency is next to *SRF* (7.3 MHz).

Experimental results

Transmission system characterization

To perform an experimental characterization of transmission system, an impedance analyzer should be used. Unfortunately we didn't have one, so we had to use alternative methods. Two different characterizations were performed. In a first case we measured every parameter of the equivalent model separately (a similar approach to theoretical characterization). The second method we could define it as an impedance matrix "rough measurement".

Let's analyze the first method. Self inductance was measured with an LCR meter (XJ2811C) and the results are shown in table 1, compared to theoretical values. For measuring mutual inductance we used the set up shown in figure 7. We measured the open-circuit voltage in one coil while the other was driven with a waveform generator (Agilent 33210A). The driven current was measured by using an external resistance. Mutual inductance can be calculated by (15). All measurements were carried out with a two channels digital oscilloscope (Agilent DSO3062A). An external capacitor was used to resonate with the coil for achieving a higher current, and therefore a higher value of V_{OC}. Resonance frequency (106 kHz) was chosen much lower than theoretical *SRF*.

$$M = \frac{|V_{OC}|}{2\pi f\left(|V_X|/R_X\right)} \tag{15}$$

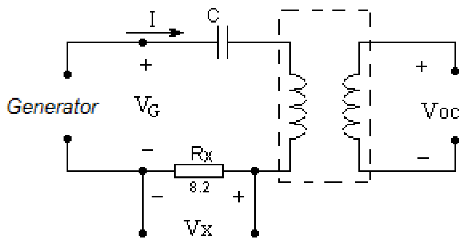

Fig. 7: Mutual inductance measurement setup.

For estimating stray capacitance and ac losses the circuit in figure 8 was used. The coil has been represented by its parallel equivalent circuit. C_{in} models oscilloscope input capacitance (13 pF). Different external capacitors (C_{ext}) were used to obtained different values of parallel resonance frequency. Total equivalent capacitance was then calculated. As

$C_T = C_P + C_{in} + C_{ext}$, C_p can be obtained by extrapolation for C_{ext} = 0.

$$C_T = \frac{1}{(2\pi f_0)^2 L} \quad (16)$$

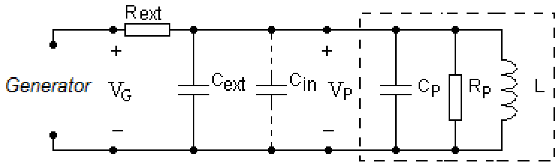

Fig. 8: Stray capacitance and ac losses measurement.

At resonance, parallel resistance (R_p) forms a resistive divider with the external resistor (R_{ext} = 121.2 kΩ), so it can be found by (17). Series resistance is obtained by (18). These values are plotted in figure 9. They are far away from theoretical values, not as the other parameters. One of the possible reasons may be that, above 3 MHz, the wave length is not much bigger than wire total length, thus radiation losses may being affecting.

$$R_P = \frac{|V_P|}{|V_G| - |V_P|} R_{ext} \quad (17)$$

$$R = \frac{(2\pi f_0 L)^2}{R_P} \quad (18)$$

Table 1: Parameters of coils model.

	Theoretical	Exp. Tx	Exp. Rx
L (µH)	70.51	69.1	68.8
C_P (pF)	6.71	6.37	6.43
R (Ω)	$0.648^* \sqrt{f\,[\text{MHz}]}$	Fig. 9	
$M_{d=r}$ (µH)	6.36	5.579	
$M_{d=2r}$ (µH)	2.22	1.997	
$M_{d=3r}$ (µH)	0.98	0.967	
$M_{d=4r}$ (µH)	0.509	0.514	

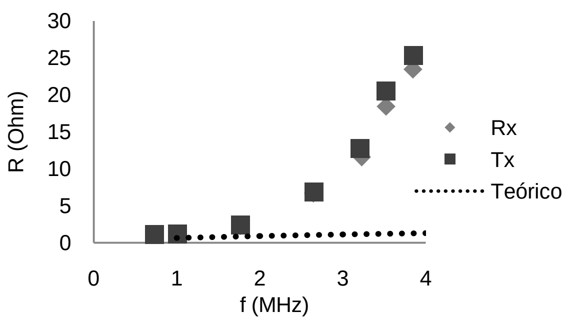

Fig. 9: Measured ac resistance.

For the z matrix rough measurement method we used the same set up of figure 7, but also measuring the waveform generator output (so an extra oscilloscope was included). Several values for the external capacitor were employed, and series resonance frequency was determined. From this frequency, external series capacitor and resistor, V_X, V_G and V_{OC}, we can reach imaginary (19) and real (20) parts of z_{11}, as well as the absolute value of z_{21} (21). Switching transmitter and receiver coils we obtain z_{22} and z_{12}. This method has two main problems: (1) no information of real and imaginary parts of mutual impedances is achieved (this will bring errors for frequencies near the SRF, where real part becomes significant due to stray capacitance effects), and (2) for high frequencies oscilloscope input capacitance can't be considered as an open-circuit.

$$\mathrm{Im}(z_{11}) = \frac{1}{2\pi f_0 C} \quad (19)$$

$$\mathrm{Re}(z_{11}) = \frac{|V_G| - |V_X|}{|V_X|} R_X \quad (20)$$

$$|z_{21}| = \frac{|V_{OC}|}{|V_X|} R_X \quad (21)$$

In figure 10 these results have been ploted (Exp. Z param.), along with z parameters obtained from theoretical equivalent model (Theo. Model) and its experimental characterization (Exp. Model). We just show values of z_{11} for $d = r$ because they didn't change significantly with distance. Both experimental characterizations exhibit similar results, with imaginary part very close to theory. Experimental real part is much bigger than theoretical prediction, as expected from ac resistance measurements. Measured $|z_{21}|$ presents a behavior that was not expected from coils model, with local minima and maxima above 3 MHz.

Output power measurement

Finally, experimental behavior of output power with different resonance frequencies was studied. We measured output voltage for three different loads. We used the same distances from transmitter to receiver that in theoretical study (10.4; 20.8, 31.2 and 41.6 cm). Eleven pairs of capacitors gave us the same number of different resonance frequencies. For each combination (132 in total) a frequency sweep was performed, for determining resonance frequency and output power. Figure 11 shows these measurements, along with predictions carried out from every characterization method (waveform generator was also characterized), for a visual comparison.

As the first important result we have the existence of a frequency that maximizes output power. This frequency depends, as expected from the theoretical study, of transmitter to receiver separation, and also of load value. This proves that increasing separation between coils

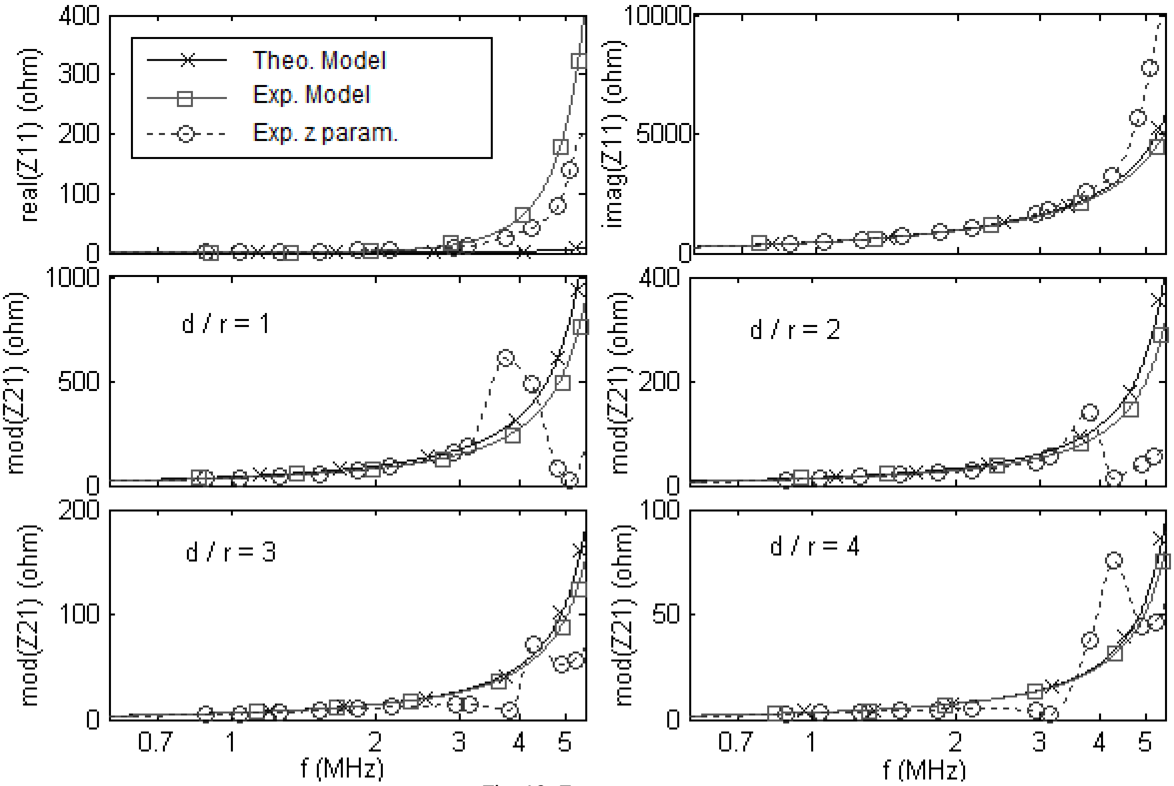

Fig. 10: Z parameters.

and load values will require a higher resonance frequency to obtain maximum output power.

For d = r we found the peaks in frequency response when resonance was set higher than its optimal value, as we saw in PSpice simulations.

We found good agreement between experimental results and predictions, for d = r and 2r (with lowest values of optimal resonance frequency). This is also a good result, because it shows that optimization can be carried out by a proper characterization of transmission system, a faster way than fully empirical optimization.

For d = 3r and d = 4r, only the prediction based on impedance matrix rough measurements was close to experimental behavior of output power, including the existence of local minimum values. We expect that a proper characterization (that is, using the proper equipment) could improve these results. By the other hand, these results also show that, in this scheme for wireless energy transfer, optimal frequency must be kept low enough for avoiding losses for radiation effects.

Conclusions

Maximum theoretical value of output power in inductively coupled resonant circuits was obtained. This value only depends of source characteristics. Optimal resonance frequency for the ideal case was also achieved. Theoretical and experimental results prove that there is a resonance frequency that maximizes output power. This frequency will be higher as distance between transmitter and receiver and also load values are increased. Optimal resonance value will be limited by coils SRF and by radiation effects. A method for finding optimal resonance frequency was developed, based on open-circuit impedance matrix characterization of transmission system.

Acknowledgements

This work was partially supported by MAEC-AECID through FORTIN project (Ref. D/024124/09), www.imse-cnm.csic.es/fortin.

References

[1] Karalis, A., J. Joannopoulos, and M. Soljačić, *Efficient wireless non-radiative mid-range energy transfer.* Annals of Physics, 2008. **323**(1): p. 34-38.

[2] Kurs, A., et al., *Wireless Power Transfer via Strongly Coupled Magnetic Resonances.* Science, 2007. **317**(83): p. 83-86.

[3] Ramo, S., J.R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*. 1965, New York: John Wiley & Sons, Inc. 754.

[4] Reitz, J.R. and F.J. Milford, *Foundations of Electromagnetic Theory*. 1959. 387.

[5] Smith, G.S., *Radiation Efficiency of Electrically Small Multiturn Loop Antennas.* IEEE Transactions on Antennas and Propagation, 1972. **AP-20**(5): p. 656-667.

[6] Yu, Q. and T.W. Holmes, *Stray Capacitance Modeling of Inductors by Using the Finite Element Method.* IEEE transactions on electromagnetic compatibility, 2001. **43**(1): p. 88-93.

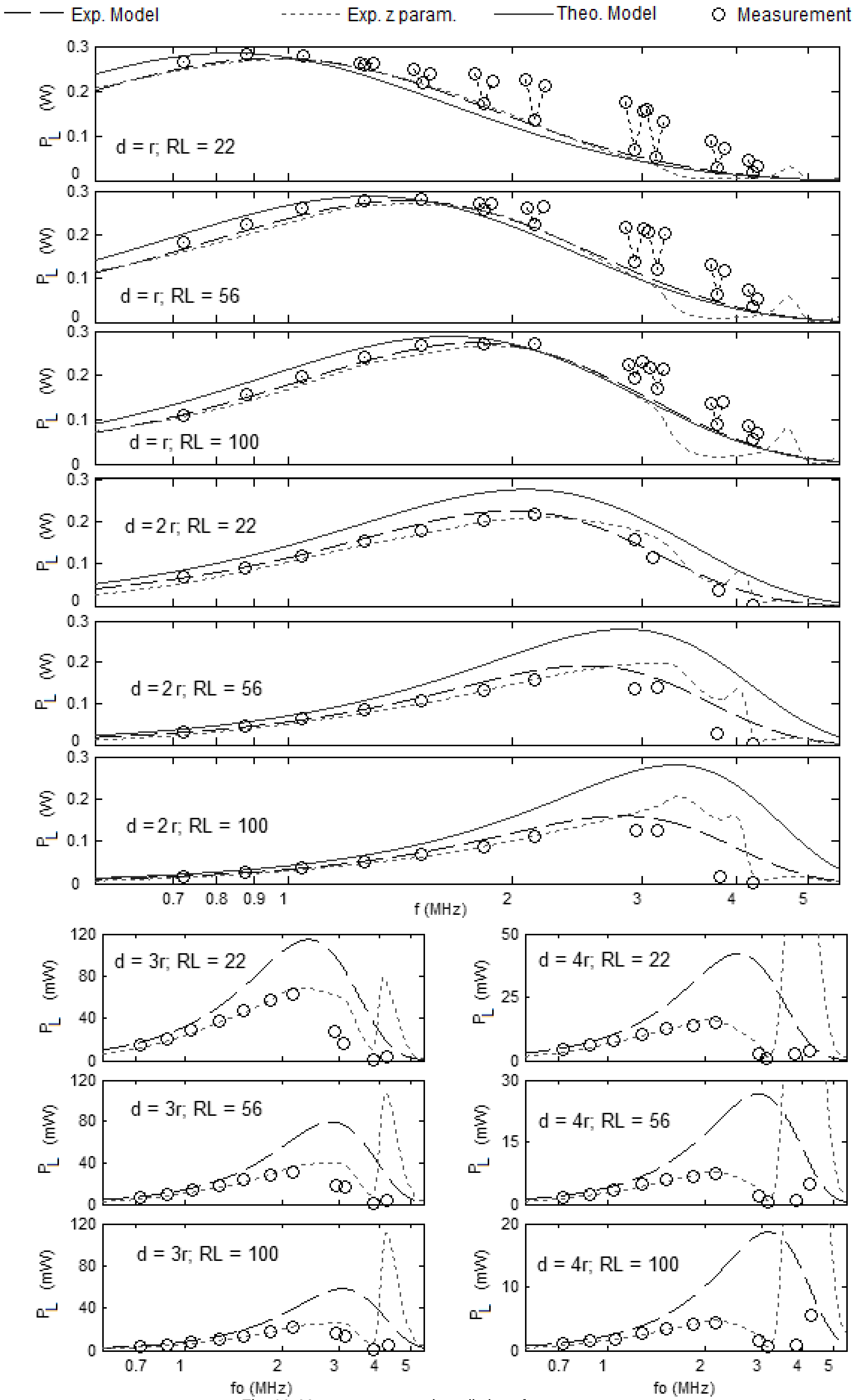

Fig. 11: Measurements and predictions for output power.