I. Basic Cosmology Elements

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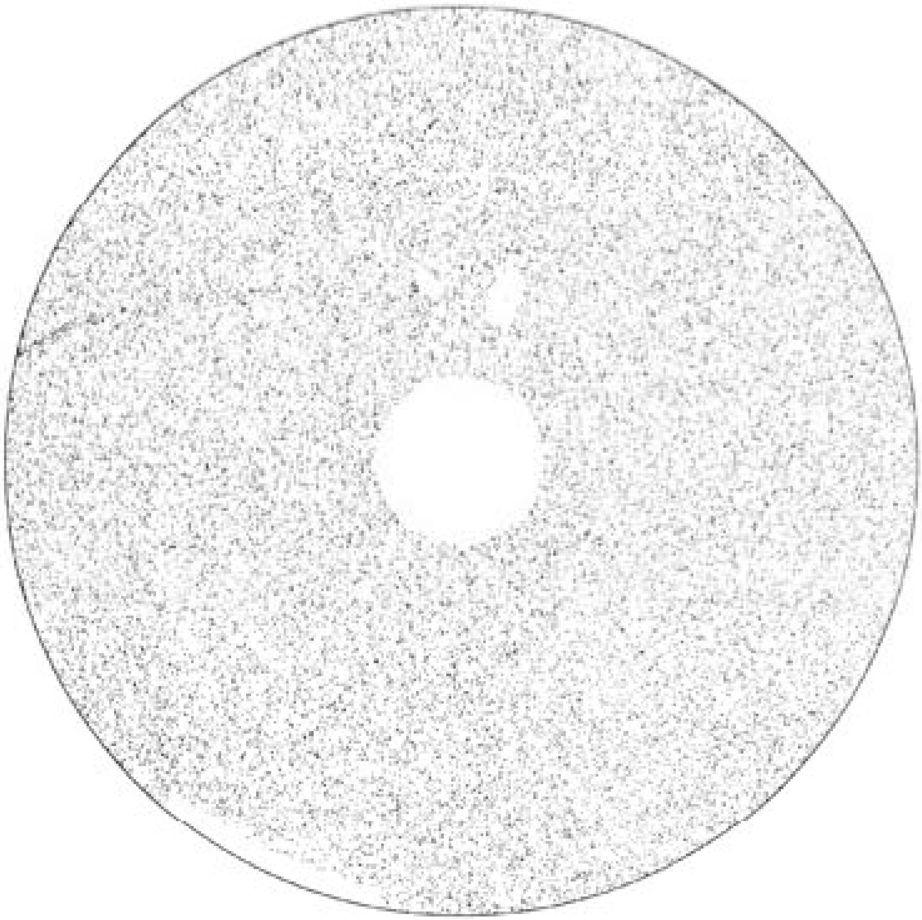
Cosmological Principle

"The Universe is homogeneous and isotropic on large-scales"

As can be seen by the position of extragalactic radio-sources

Angular distribution of the ~ 31 000 brightest 6cm radio sources in the sky

(Peebles 1993)

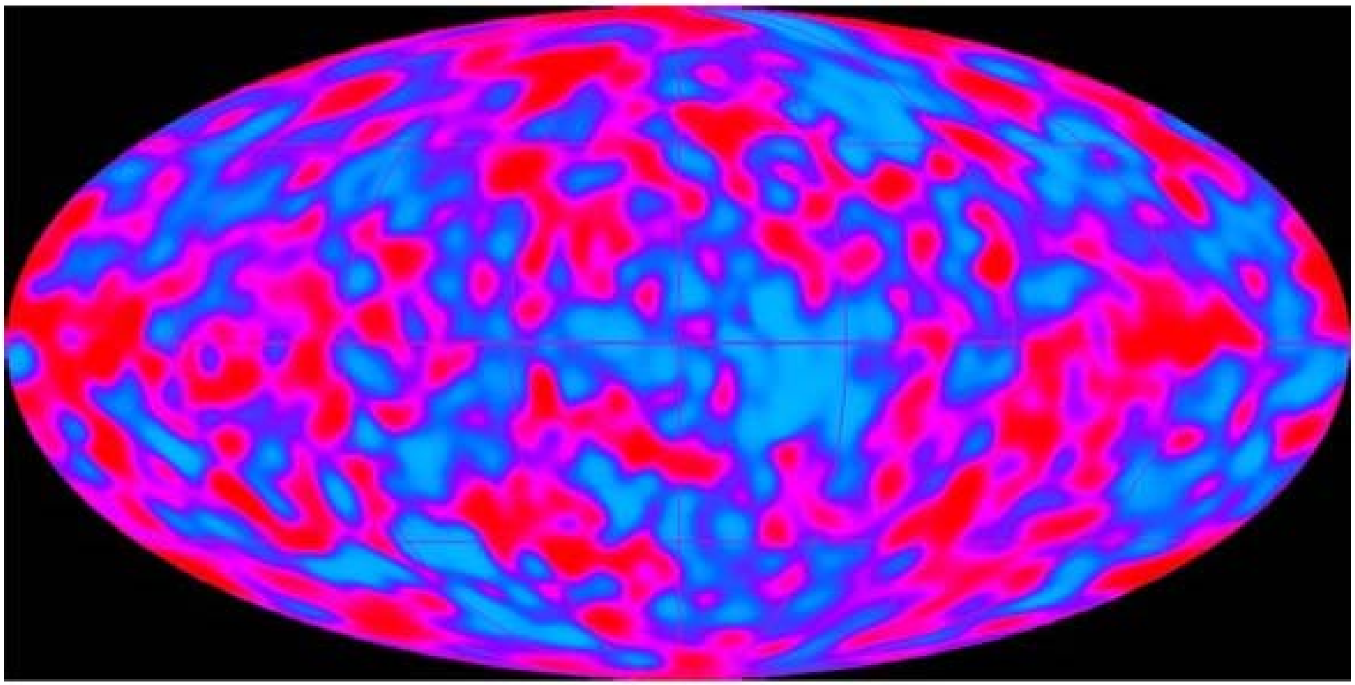


(From R. Bender's notes)

Cosmological Principle

"The Universe is homogeneous and isotropic on large-scales"

As can be seen by Cosmic Microwave Background (CMB) radiation



Temperature fluctuations in the Cosmic Microwave Background as measured by the COBE satellite. The amplitude of the fluctuations is only $\Delta T/T \simeq 10^{-5}$ and reflects density inhomogeneities in the baryons of the same order about 370000 years after the big bang.

(From R. Bender's notes)

Cosmological Principle

“The Universe is homogeneous and isotropic on large-scales”

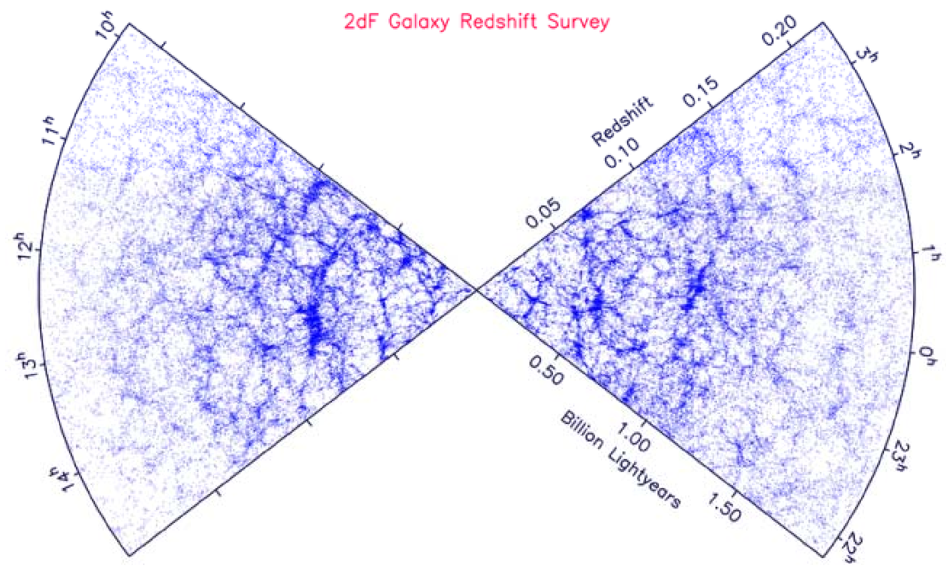
As can be seen by the 2-point correlation function of galaxies, which are clustered in scales of few x h^{-1} Mpc.

Other LSS scales: supercluster associations ~ 100 h^{-1} Mpc

filaments ~ 100 –250 h^{-1} Mpc

voids ~ 60 h^{-1} Mpc

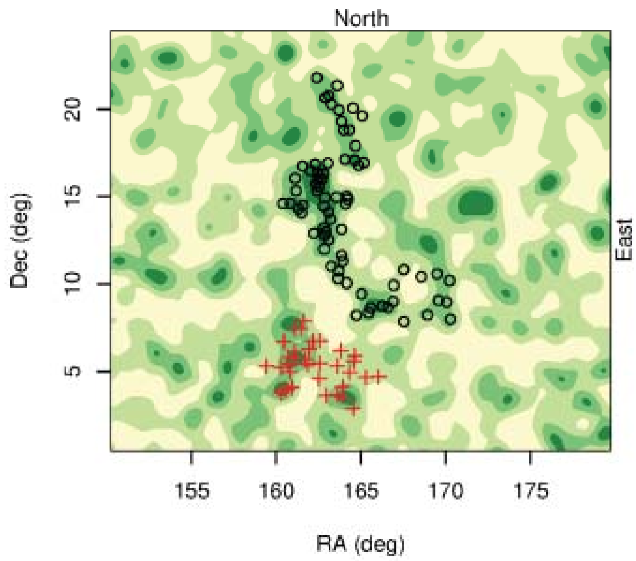
There is a characteristic scale 300 h^{-1} Mpc $\leq l \leq cH_0^{-1}$ averaged over which the Universe can be considered homogeneous.

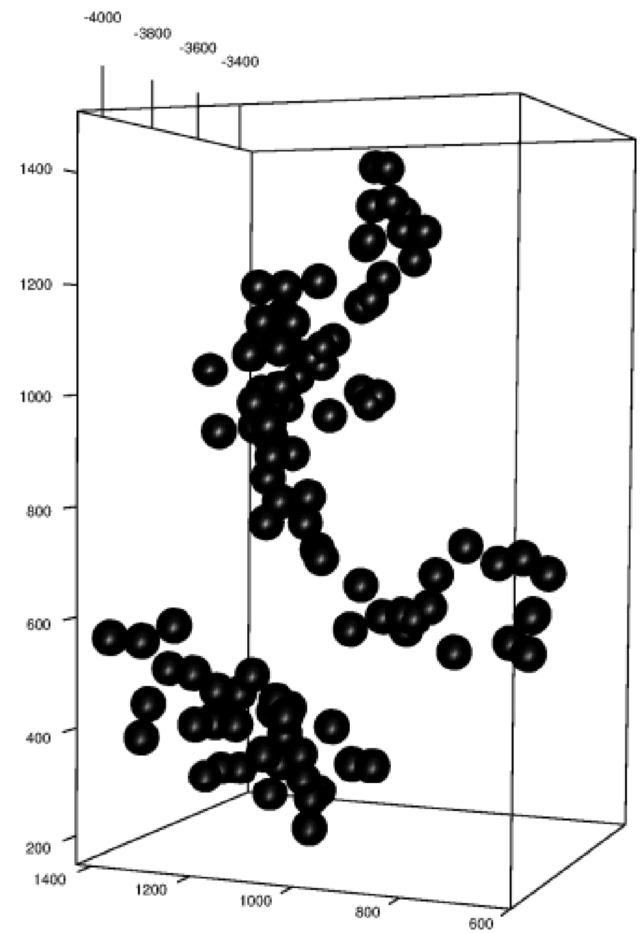


Cosmological Principle

"The Universe is homogeneous and isotropic on large-scales"

But there are a few large-scale structures in the Universe that are posed as potential problems of anisotripies at ≥ 500 Mpc (comoving scale): e.g. Huge – Large Quasar Group at $1.17 < z < 1.42$, composed of 73 QSOs in a 1240 x 640 x 370 Mpc structure (Clowes et al. 2013, MNRAS). However, detailed 3D calculations out of sphericity are difficult to carry out nowadays to assess compatibility within a Gaussian primordial field fluctuation analysis (Sheth + Diaferio 2011 MNRAS)





The original Hubble diagram

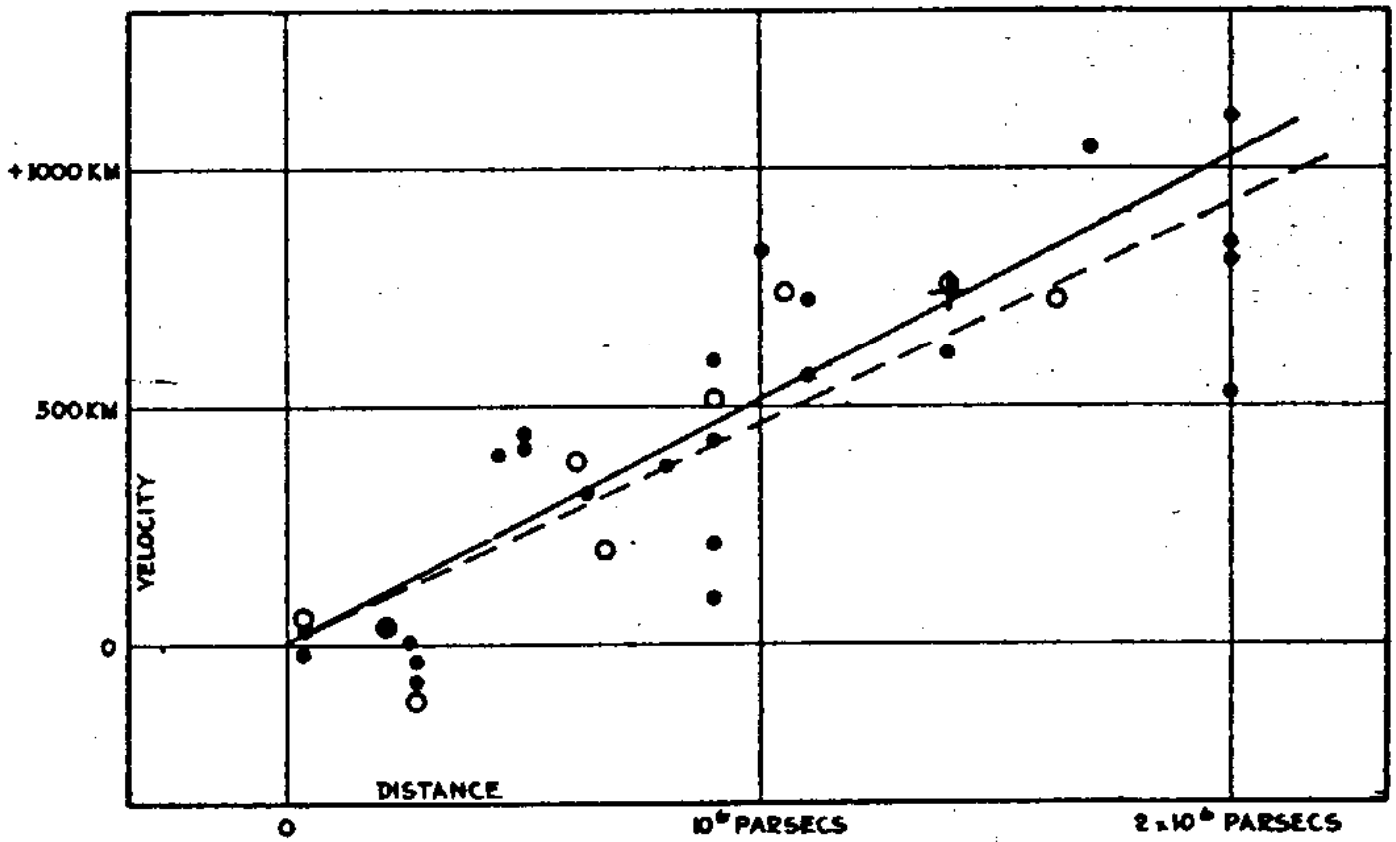


FIGURE 1

Hubble (1929) in Proceedings of the National Academy of Sciences,
Lemaître had done a previous (1927) estimation of H_0 based on Hubble´s data

The original Hubble diagram

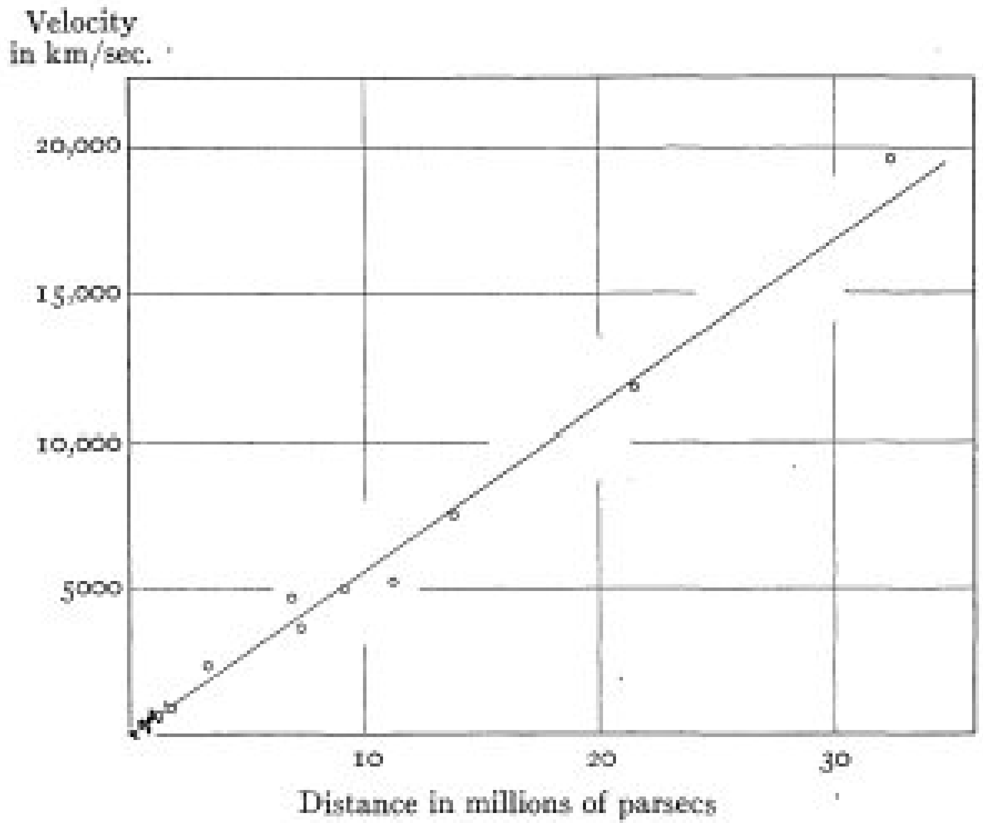
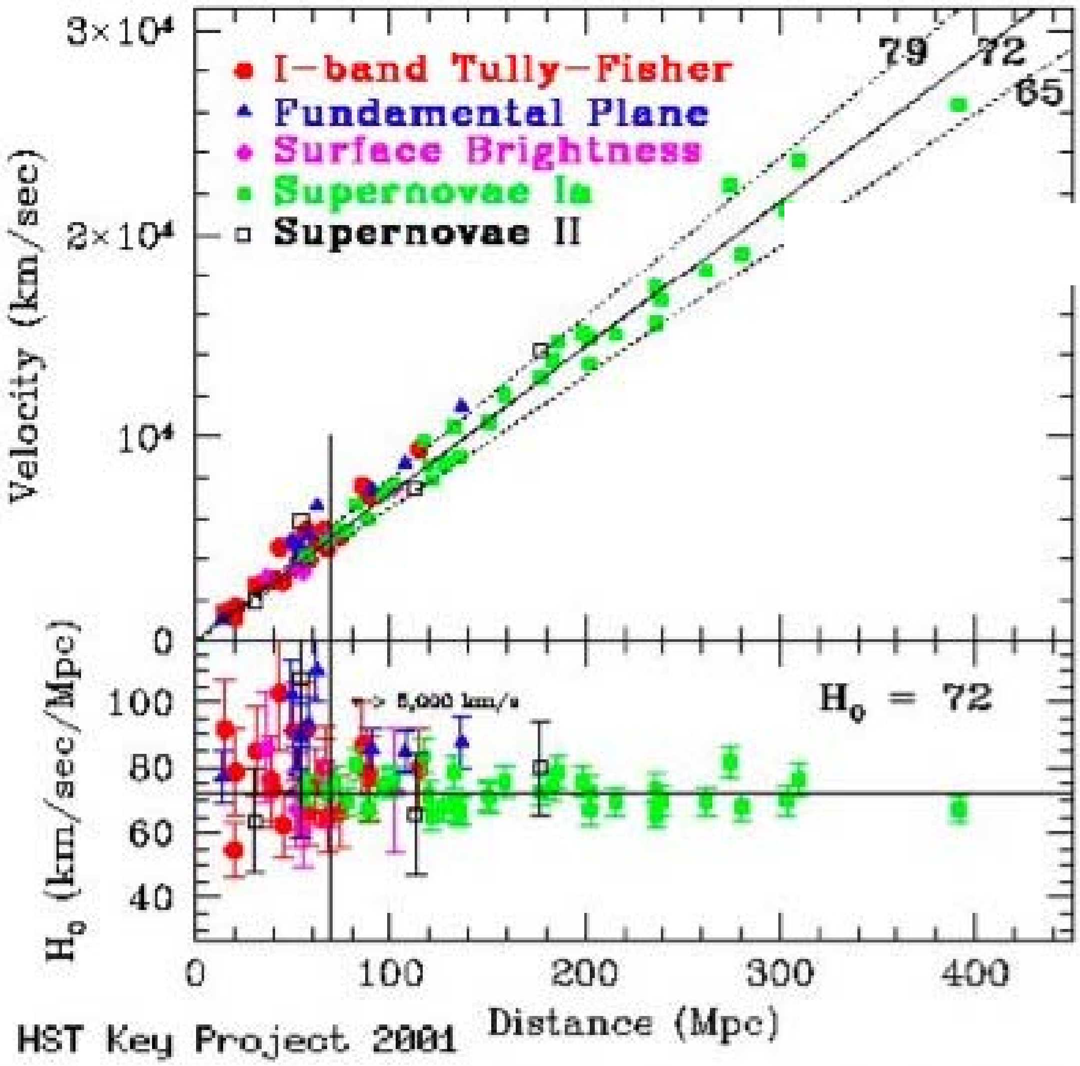


FIG. 5.—The velocity-distance relation. The circles represent mean values for clusters or groups of nebulae. The dots near the origin represent individual nebulae, which, together with the groups indicated by the lowest two circles, were used in the first formulation of the velocity-distance relation.

Hubble & Humason (1931) Astrophysical Journal

The value of H_0:

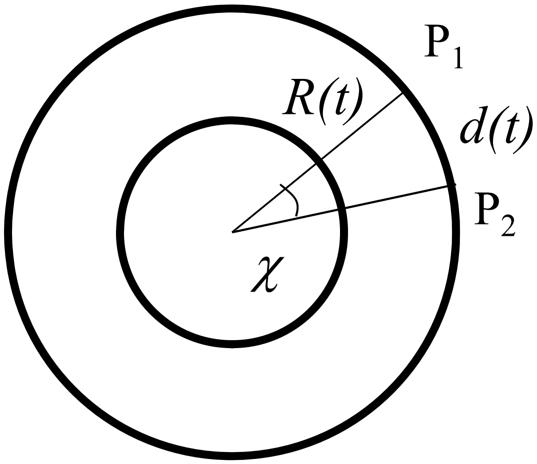
H_0=72±8 km/s/Mpc (Freedman et al. 2001, ApJ)



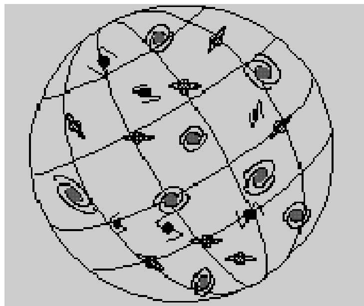
The origin of the Hubble "constant"

Can be deduced from an expanding homogeneous universe.

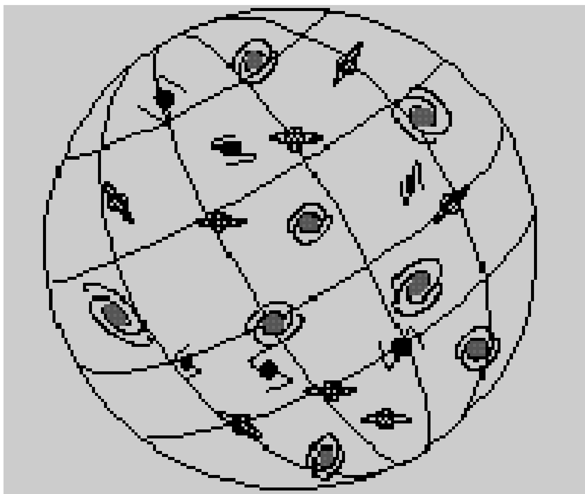
Let's imagine a 1D Universe, on an expanding circle:
d(t) is the proper distance between two points P_1P_2
R(t) is the scale factor or growth
χ is a comoving coordinate, that defines the distance between P_1P_2 (comoving with the expanding universe)



$$\left.\begin{array}{l} d = R\chi \Rightarrow v = \dot{d} = \dot{R}\chi = \dfrac{\dot{R}}{R}d \\ v = H_0 d \text{ in general } v = Hd \end{array}\right\} \Rightarrow \boxed{H = \frac{\dot{R}}{R}}$$



$\Rightarrow$



The Friedmann-Lemaître-Robertson-Walker metric (1922-1936)

$$\boxed{ds^2 = c^2dt^2 - R^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]}$$

where *(r, θ, φ)* are spherical comoving coordinates, *R* is the scale factor, and *k* is a constant related to the curvature.

It can be deduced purely from symmetry alone for a homogeneous universe

For a 2D universe on the surface of a sphere, the proper distance P_1P_2

$$(dl)^2 = (Rd\theta)^2 + (bd\phi)^2 = \left(\frac{db}{\sqrt{1-b^2/R^2}}\right)^2 + (bd\phi)^2$$

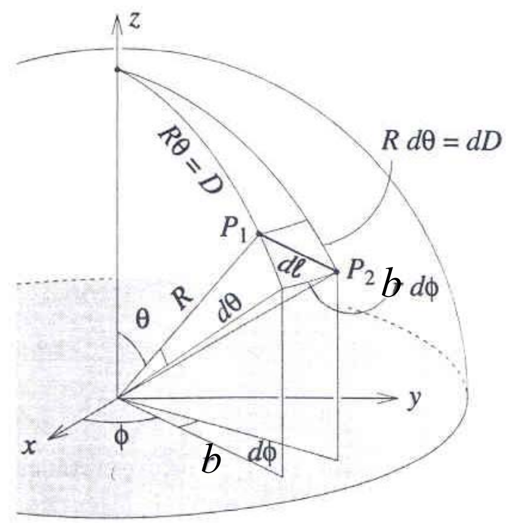
where $K=1/R^2$ is the curvature at *t*

For a 3D universe

$$(dl)^2 = \left(\frac{db}{\sqrt{1-Kb^2}}\right)^2 + (bd\theta)^2 + (b\sin\theta d\phi)^2$$

For space-time, introducing a time-independent curvature $K \equiv k/R^2$, and the comoving coordinate *r*, such that $b=Rr$, the geodesic is given by

$$(ds)^2 = (cdt)^2 - (dl)^2 = (cdt)^2 - R^2\left[\left(\frac{dr}{\sqrt{1-kr^2}}\right)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2\right]$$



(b)

(following Carroll & Ostlie's "Modern Astrophysics", Addison-Wesley)

The cosmological origin of redshift

$z \equiv (\lambda - \lambda_{rep})/\lambda_{rep}$

It can be deduced from the FLRW metric

Light travels along null geodesics $ds=0$. If we follow the path of light from r_1 to $r=0$, the null geodesic follows constant (θ, ϕ), and $d\theta = d\phi = 0$.

Hence, the RW metric $\Rightarrow$ $\dfrac{cdt}{R} = \pm \dfrac{dr}{\sqrt{1-kr^2}}$

Two consecutive crests leave at t_1 and $t_1+\Delta t_1$ and are received at t_0 and $t_0+\Delta t_0$

$$\int_{t_1}^{t_0} \frac{cdt}{R(t)} = -\int_{r_1}^{0} \frac{dr}{\sqrt{1-kr^2}} = \int_{0}^{r_1} \frac{dr}{\sqrt{1-kr^2}} \tag{4}$$

A successive crest, leaving $\mathcal{L}_1$ at $t_1 + \delta t_1$ will arrive at $r = 0$ at $t_0 + \delta t_0$, but since we have constant radial coordinates:

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{cdt}{R(t)} = \int_{0}^{r_1} \frac{dr}{\sqrt{1-kr^2}} \Longrightarrow \int_{t_1}^{t_0} \frac{cdt}{R(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{cdt}{R(t)} \tag{5}$$

and rearranging the limits of integration:

$$\int_{t_1}^{t_1+\delta t_1} + \int_{t_1+\delta t_1}^{t_0} = \int_{t_1+\delta t_1}^{t_0} + \int_{t_0}^{t_0+\delta t_0}$$

we get that:

$$\int_{t_1}^{t_1+\delta t_1} \frac{cdt}{R(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{cdt}{R(t)}$$

Now if $\delta t \ll t$ we can consider $R(t)$ constant over the integration time and therefore $\delta t_1/R(t_1) = \delta t_0/R(t_0)$ and since $\delta t_{1,0}$ is the time between successive wave crests of the emitted (detected) light, it is also the wavelength of the emitted (detected) light:

$$\frac{\lambda_1}{\lambda_0} = \frac{R(t_1)}{R(t_0)}$$

and the *Cosmological* redshift, z is defined as the ratio of the detected wavelength to that emitted:

$$\boxed{1 + z = \frac{\lambda_0}{\lambda_1} = \frac{R(t_0)}{R(t_1)}} \tag{6}$$

The cosmological origin of redshift

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Light travels along null geodesics $ds=0$. If we follow the path of light from r_1 to $r=0$, the null geodesic follows constant (θ, ϕ), and $d\theta = d\phi = 0$.

Hence, the RW metric $\Rightarrow$ $\frac{cdt}{R} = \pm \frac{dr}{\sqrt{1-kr^2}}$ $\Rightarrow$ $1+z = \frac{\lambda_0}{\lambda_1} = \frac{R(t_0)}{R(t_1)} \Rightarrow R \propto (1+z)^{-1}$

It is not a Doppler effect, but rather a property of the expanding non-Euclidean space-time.

The wavelength of light shifts to the red $\lambda \propto R(t)$

The energy carried by the wave decreases as the Universe expands $E=hc/\lambda \propto 1/R(t)$

In general, every single quantity has to be converted

You might find $z \geq \sim 1$ interpreted as recession velocity, using the relativistic Doppler effect formula:

$$1+z = \sqrt{\frac{1+v/c}{1-v/c}}$$

Friedmann's Equation (1922)

$$\boxed{\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2}$$

Although it was deduced from Einstein's field equations, it can also be deduced from Newtonian gravity.

Consider a sphere about some arbitrary point, such that the radius is *R(t)*.

homogeneity $\Rightarrow$ $\nabla\rho=0$

Isotropy $\Rightarrow$ $\nabla.\vec{v} = 3H = 3\frac{\dot{R}}{R}$

From Newton's equation of motion

$$\ddot{R} = -GM/R^2 \Rightarrow \dot{R}\ddot{R} = -GM\dot{R}/R^2 \Rightarrow \frac{d}{dt}\left[\frac{1}{2}\dot{R}^2 - \frac{GM}{R}\right] = 0 \Rightarrow$$

$$\frac{1}{2}\dot{R}^2 - \frac{4}{3}\pi G\rho R^2 = C \Rightarrow \dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2$$

where ρ has contributions from matter, radiation and vacuum energy

$$\boxed{\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 - \frac{\Lambda c^2}{3}R^2 = -kc^2}$$

where ρ has only contributions due to matter and radiation

(following M. Plionis' notes or Peacock 1999)

Friedmann's Acceleration Equation (1922)

$$\boxed{\ddot{R} = -\frac{4\pi G}{3} R(\rho + 3p/c^2)}$$

Deriving $\qquad \dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2$

Conservation of energy $\qquad d(\rho c^2 R^3) = -p\,d(R^3)$

$$\Rightarrow \ddot{R} = -\frac{4\pi G}{3} R(\rho + 3p/c^2)$$

$$\boxed{\ddot{R} = -\frac{4\pi G}{3} R(\rho + 3p/c^2) + \frac{\Lambda c^2}{3}}$$

where ρ has only contributions due to matter and radiation

(following M. Plionis' notes or Peacock 1999)

Friedmann's Equation (1922)

General Relativity

The basic equations of **General Relativity** are **Einstein's Field Equations**:

$$\boxed{R_{ij} - \frac{1}{2} g_{ij} \mathcal{R} = 8\pi G T_{ij} + \Lambda g_{ij}} \qquad (13.1)$$

R_{ij}:	Ricci tensor ($R_{ij} = R_{ij}(g_{ij})$)	$\leftrightarrow$ space–time curvature
g_{ij}:	metric tensor	$\leftrightarrow$ space–time distances $ds^2 = g_{ij} dx^i dx^j$
$\mathcal{R}$:	Ricci scalar ($\mathcal{R} = g^{ik} R_{ik}$)	$\leftrightarrow$ space–time curvature
G:	gravitational constant	
T_{ij}:	energy–momentum tensor	$\leftrightarrow$ mass, energy, . . .
Λ:	cosmological constant	

The Field Equations connect the energy (and thus mass) distribution in space to its geometrical properties (curvature).

For details see e.g. Weinberg, *Gravitation and Cosmology*, J. Wiley 1972, or Misner, Thorne, & Wheeler, *Gravitation*, Freeman 1970.

(from R. Bender's notes)

Friedmann's Equation (1922)

13.3 The Friedmann Equations

The geometry of a homogeneous and isotropic universe is described by the g_{ij} of the Robertson–Walker metric (13.2). In order to obtain a solution for the *dynamics* of the universe, the Ricci tensor needs to be calculated from the g_{ij} and the field equations have to be solved for an energy momentum tensor reflecting a homogeneous distribution of mass. For a perfect homogeneous fluid T_{ij} takes the simple form:

$$T_{ij} = \frac{1}{c^2} \begin{pmatrix} \varrho c^2 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

with the density ϱ and the pressure p.

(from R. Bender's notes)

Friedmann's Equation (1922)

Inserting g_{ij}, R_{ij} and T_{ij} in the field equations (13.1) yields the two **Friedmann equations**:

$$\boxed{\ddot{R} = -\frac{4\pi G\,R}{3}\left(\varrho + 3\frac{p}{c}\right) + \frac{1}{3}\Lambda\,R} \tag{13.5}$$

$$\boxed{\dot{R}^2 = \frac{8\pi G\,\varrho}{3}R^2 + \frac{1}{3}\Lambda\,R^2 - \frac{c^2}{R_{c,0}^2}} \tag{13.6}$$

These equations govern the dynamical evolution of the universe (i.e. the time evolution of the scale factor $R(t)$). The Friedmann equations connect this evolution to the intrinsic properties (density ϱ, pressure p, cosmological constant Λ, curvature radius $R_{c,0}$ today) of the universe.

(from R. Bender's notes)

Cosmological Parameters 1

For a flat Universe $k=0$ with no cosmological parameter $\Lambda=0$

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 - \frac{\Lambda c^2}{3} R^2 = -kc^2 \Rightarrow \left(H^2 - \frac{8\pi G}{3}\rho \right) R^2 = 0$$

$$\boxed{\rho_c = \frac{3H_0^{\ 2}}{8\pi G} = 1.88 \times 10^{-29} h^2 \text{ gcm}^{-3}}$$

where $h \equiv H_0/100$ km/s/Mpc. The critical density is the density necessary to have a flat Universe.

The density of the Universe is often expressed as the density parameter

$$\boxed{\Omega = \frac{\rho}{\rho_c}}$$

(following M. Plionis' notes or Peacock 1999)

Einstein-de Sitter Universe (1932)

This is a universe with $\Omega_m = 1$, $\Omega_\Lambda = 0$, i.e. the universe is Euclidean:

$$\dot{R}^2 = \frac{8\pi G\varrho}{3} R^2$$

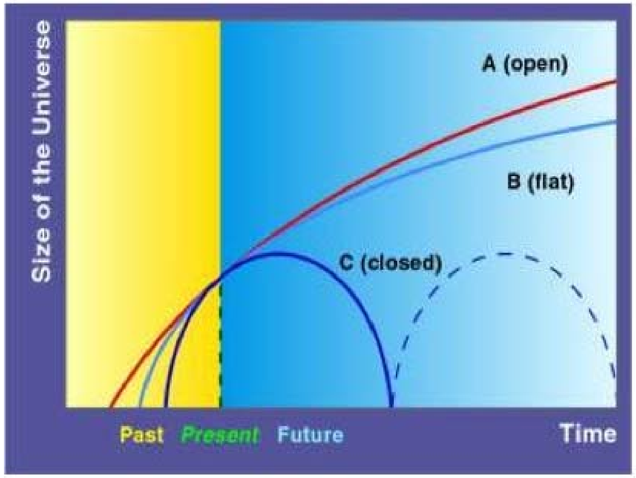
which can be integrated and yields:

$$R^{1/2}\, dR = \left(\frac{8\pi G\varrho_0}{3}\right)^{1/2} dt \qquad \rho = \rho_0 R^{-3}$$

Using the definition of Ω_m (13.8) and considering that we assumed $\Omega_m = 1$, we have $H_0^2 = (8\pi G\varrho_0)/3$ and thus:

$$\boxed{R = \left(\frac{3}{2} H_0\, t\right)^{2/3} \qquad (p = 0, \Lambda = 0, \Omega_m = 1)}$$

R



$R \propto t^{2/3}$

(R. Benders' notes)

Equations of state

$$\boxed{\rho \propto R^{-3(1+w)}}$$

In general $p=w<v^2>\rho$

For a matter dominated universe: $\rho \propto R^{-3}$, $p=0$, $w=0$ (dust approximation)

For a radiation dominated universe (photons have the E reduced by R^{-1}):
$\rho \propto R^{-4}$, $p=1/3\ \rho c^2$, $w=1/3$

For a vacuum dominated universe ρ=constant, $w=-1$

Friedmann's Equation rewritten: parameters 2

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2 \Rightarrow \left(H^2 - \frac{8\pi G}{3}\rho\right)R^2 = -kc^2$$

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 - \frac{\Lambda c^2}{3}R^2 = -kc^2 \Rightarrow \frac{\dot{R}}{R} = H_0\left[\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_k(1+z)^2 + \Omega_\Lambda\right]^{1/2}$$

where $\Omega_\Lambda \equiv \dfrac{\Lambda c^2}{3H_0{}^2}$, $\Omega_k \equiv \dfrac{-kc^2}{R_o^2 H_o^2}$

$$\boxed{H(z) = H_0 E(z)}$$

and $\Omega_\Lambda + \Omega_k + \Omega_m + \Omega_r = 1$

(following M. Plionis' notes or Peacock 1999)

Cosmological Parameters 3

$$\boxed{q \equiv -\frac{\ddot{R}R}{\dot{R}^2}}$$

From a Taylor's expansion

$$R(t) = R_0 + \dot{R}_0(t - t_0) + \frac{1}{2}\ddot{R}_0(t - t_0)^2 + \ldots$$

$$R(t)/R_0 = 1 + H_0(t - t_0) - \frac{q_0}{2}H_0{}^2(t - t_0)^2 + \ldots$$

For a matter dominated *Λ=0* universe the deceleration constant is another classical cosmological parameter.

$$\ddot{R} = -\frac{4\pi G}{3}R(\rho + 3p/c^2) \Rightarrow q_0 = \Omega_m/2$$

(following M. Plionis' notes or Peacock 1999)

The age of the Universe

Using $H(z) = H_0 E(z)$ at the present epoch, we have $\dot{R}/R_\circ = H_\circ E(z)/(1+z)$ and from

$$R \propto (1+z)^{-1} \Rightarrow \quad \mathrm{d}R/R_\circ = -\mathrm{d}z/(1+z)^2 \quad \Rightarrow dt = \frac{-1}{H_0}\frac{dz}{E(z)(1+z)} \tag{25}$$

we obtain the age of the Universe:

$$\boxed{t_\circ = \frac{1}{H_\circ}\int_0^\infty \frac{\mathrm{d}z}{(1+z)E(z)}} \tag{26}$$

For example, in an *Einstein-de Sitter* universe ($\Omega_\Lambda = \Omega_\mathrm{k} = 0$) we have:

$$t_\circ = \frac{2}{3H_\circ} \tag{27}$$

while for a $\Omega_\Lambda > 0$ model we obtain:

$$t_\circ^\Lambda = \frac{2}{3H_\circ}\frac{1}{\sqrt{\Omega_\Lambda}}\sinh^{-1}\left[\sqrt{\frac{\Omega_\Lambda}{\Omega_\mathrm{m}}}\right] \tag{28}$$

We therefore see that if $\Omega_\Lambda > 0$ we have that the age of the Universe is larger than what is predicted in an Einstein-de Sitter Universe.

(following M. Plionis' notes)

The age of the Universe

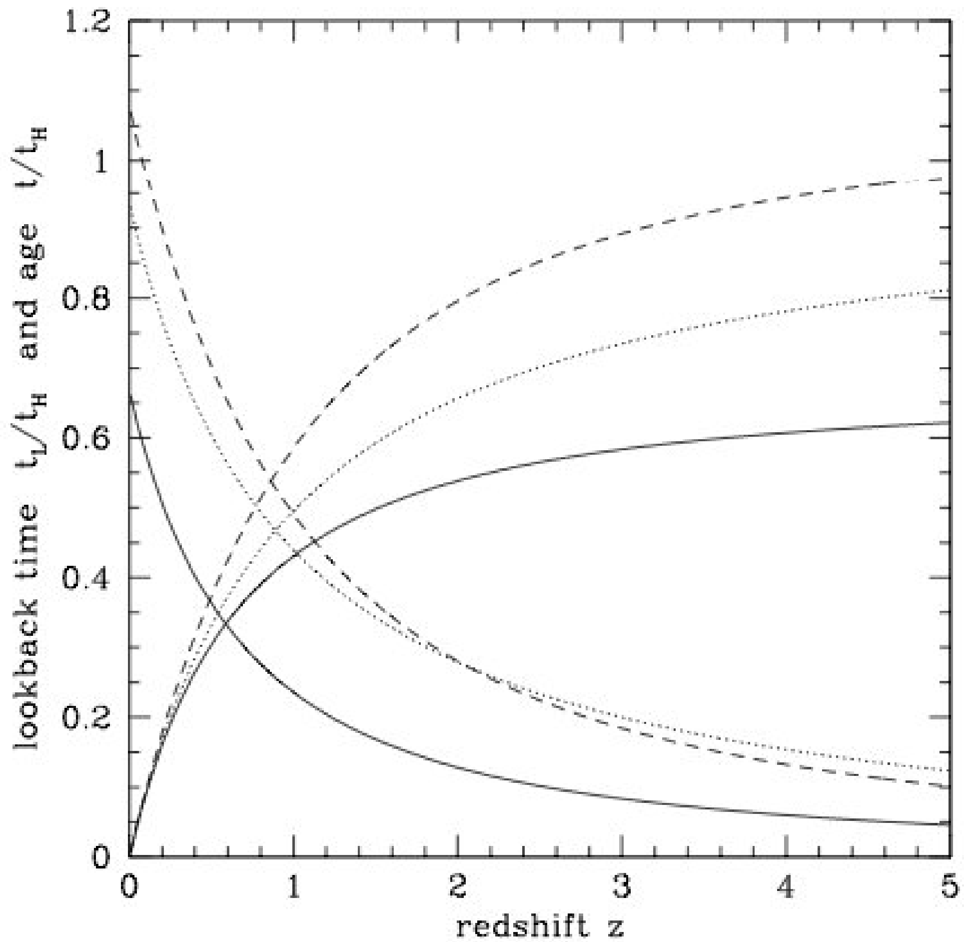
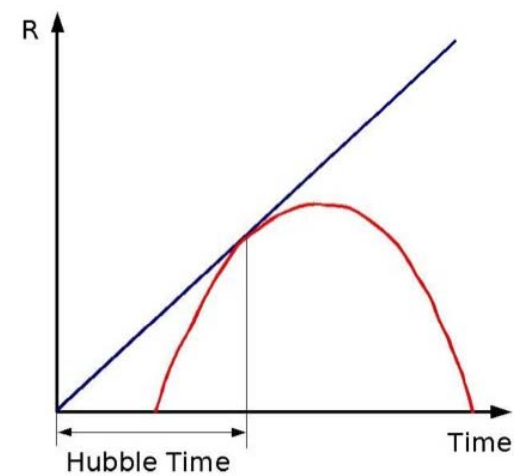


Figure 6: The dimensionless lookback time t_L/t_H and age t/t_H. Curves cross at the redshift at which the Universe is half its present age. The three curves are for the three world models, $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; $(0.05, 0)$, dotted; and $(0.2, 0.8)$, dashed.

$t_H \equiv 1/H_0$ Hubble time =

$3.09\times10^{17}\, h^{-1}$ s =
$9.80\times10^{9}\, h^{-1}$ yr



$$t_L = t_H \int \frac{dz}{(1+z)E(z)}$$

(Hogg 1999, astro-ph/9905116)