Friedmann's Equation (1922)

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2$$

Although it was deduced from Einstein's field equations, it can also be deduced from Newtonian gravity.

Consider a sphere about some arbitrary point, such that the radius is *R(t)*.

homogeneity ⇒ $\nabla\rho=0$

Isotropy ⇒ $\nabla.\vec{v} = 3H = 3\frac{\dot{R}}{R}$

From Newton's equation of motion

$$\ddot{R} = -GM/R^2 \Rightarrow \dot{R}\ddot{R} = -GM\dot{R}/R^2 \Rightarrow \frac{d}{dt}\left[\frac{1}{2}\dot{R}^2 - \frac{GM}{R}\right] = 0 \Rightarrow$$

$$\frac{1}{2}\dot{R}^2 - \frac{4}{3}\pi G\rho R^2 = C \Rightarrow \dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2$$

where ρ has contributions from matter, radiation and vacuum energy

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 - \frac{\Lambda c^2}{3}R^2 = -kc^2$$

where ρ has only contributions due to matter and radiation

(following M. Plionis' notes or Peacock 1999)

Friedmann's Acceleration Equation (1922)

$$\ddot{R} = -\frac{4\pi G}{3} R(\rho + 3p/c^2)$$

Deriving $\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2$

Conservation of energy $d(\rho c^2 R^3) = -p d(R^3)$

$$\Rightarrow \ddot{R} = -\frac{4\pi G}{3} R(\rho + 3p/c^2)$$

$$\ddot{R} = -\frac{4\pi G}{3} R(\rho + 3p/c^2) + \frac{\Lambda c^2}{3}$$

where ρ has only contributions due to matter and radiation

(following M. Plionis' notes or Peacock 1999)

Friedmann's Equation (1922)

General Relativity

The basic equations of **General Relativity** are **Einstein's Field Equations**:

$$\boxed{R_{ij} - \frac{1}{2} g_{ij} \mathcal{R} = 8\pi G T_{ij} + \Lambda g_{ij}} \qquad (13.1)$$

R_{ij}: Ricci tensor ($R_{ij} = R_{ij}(g_{ij})$) $\leftrightarrow$ space–time curvature
g_{ij}: metric tensor $\leftrightarrow$ space–time distances $ds^2 = g_{ij} dx^i dx^j$
$\mathcal{R}$: Ricci scalar ($\mathcal{R} = g^{ik} R_{ik}$) $\leftrightarrow$ space–time curvature
G: gravitational constant
T_{ij}: energy–momentum tensor $\leftrightarrow$ mass, energy, ...
Λ: cosmological constant

The Field Equations connect the energy (and thus mass) distribution in space to its geometrical properties (curvature).

For details see e.g. Weinberg, *Gravitation and Cosmology*, J. Wiley 1972, or Misner, Thorne, & Wheeler, *Gravitation*, Freeman 1970.

(from R. Bender's notes)

Friedmann's Equation (1922)

13.3 The Friedmann Equations

The geometry of a homogeneous and isotropic universe is described by the g_{ij} of the Robertson–Walker metric (13.2). In order to obtain a solution for the *dynamics* of the universe, the Ricci tensor needs to be calculated from the g_{ij} and the field equations have to be solved for an energy momentum tensor reflecting a homogeneous distribution of mass. For a perfect homogeneous fluid T_{ij} takes the simple form:

$$T_{ij} = \frac{1}{c^2} \begin{pmatrix} \varrho c^2 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

with the density ϱ and the pressure p.

(from R. Bender's notes)

Friedmann's Equation (1922)

Inserting g_{ij}, R_{ij} and T_{ij} in the field equations (13.1) yields the two **Friedmann equations**:

$$\ddot{R} = -\frac{4\pi G\, R}{3}\left(\varrho + 3\frac{p}{c}\right) + \frac{1}{3}\Lambda R \tag{13.5}$$

$$\dot{R}^2 = \frac{8\pi G\, \varrho}{3} R^2 + \frac{1}{3}\Lambda R^2 - \frac{c^2}{R_{c,0}^2} \tag{13.6}$$

These equations govern the dynamical evolution of the universe (i.e. the time evolution of the scale factor $R(t)$). The Friedmann equations connect this evolution to the intrinsic properties (density ϱ, pressure p, cosmological constant Λ, curvature radius $R_{c,0}$ today) of the universe.

(from R. Bender's notes)

Cosmological Parameters 1

For a flat Universe $k=0$ with no cosmological parameter $\Lambda=0$

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 - \frac{\Lambda c^2}{3}R^2 = -kc^2 \Rightarrow \left(H^2 - \frac{8\pi G}{3}\rho\right)R^2 = 0$$

$$\rho_c = \frac{3H_0{}^2}{8\pi G} = 1.88 \times 10^{-29} h^2 \text{ gcm}^{-3}$$

where $h \equiv H_0/100$ km/s/Mpc. The critical density is the density necessary to have a flat Universe.

The density of the Universe is often expressed as the density parameter

$$\Omega = \frac{\rho}{\rho_c}$$

(following M. Plionis' notes or Peacock 1999)

Einstein-de Sitter Universe (1932)

This is a universe with $\Omega_m = 1$, $\Omega_\Lambda = 0$, i.e. the universe is Euclidean:

$$\dot{R}^2 = \frac{8\pi G\varrho}{3} R^2$$

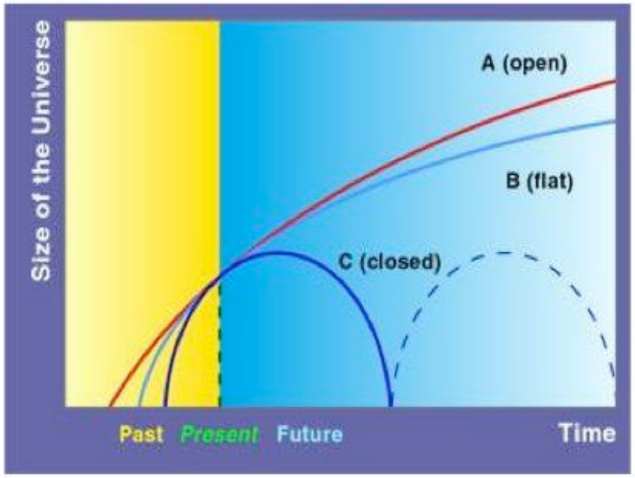
which can be integrated and yields:

$$R^{1/2}\, dR = \left(\frac{8\pi G\varrho_0}{3}\right)^{1/2} dt \qquad \rho = \rho_0 R^{-3}$$

Using the definition of Ω_m (13.8) and considering that we assumed $\Omega_m = 1$, we have $H_0^2 = (8\pi G\varrho_0)/3$ and thus:

$$R = \left(\frac{3}{2} H_0\, t\right)^{2/3} \qquad (p = 0, \Lambda = 0, \Omega_m = 1)$$

R



$R \propto t^{2/3}$

(R. Benders' notes)

Equations of state

$$\rho \propto R^{-3(1+w)}$$

In general $p=w<v^2>\rho$

For a matter dominated universe: $\rho \propto R^{-3}$, $p=0$, $w=0$ (dust approximation)

For a radiation dominated universe (photons have the E reduced by R^{-1}):
$\rho \propto R^{-4}$, $p=1/3\ \rho c^2$, $w=1/3$

For a vacuum dominated universe ρ=constant, $w=-1$

Friedmann's Equation rewritten: parameters 2

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2 \Rightarrow \left(H^2 - \frac{8\pi G}{3}\rho\right)R^2 = -kc^2/R^2$$

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 - \frac{\Lambda c^2}{3}R^2 = -kc^2 \Rightarrow \frac{\dot{R}}{R} = H_0\left[\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_k(1+z)^2 + \Omega_\Lambda\right]^{1/2}$$

where $\Omega_\Lambda \equiv \frac{\Lambda c^2}{3H_0^2}$, $\Omega_k \equiv \frac{-kc^2}{R_o^2 H_o^2}$

$$H(z) = H_0 E(z)$$

and $\Omega_\Lambda + \Omega_k + \Omega_m + \Omega_r = 1$

(following M. Plionis' notes or Peacock 1999)

Cosmological Parameters 3

$$q \equiv -\frac{\ddot{R}R}{\dot{R}^2}$$

From a Taylor's expansion

$$R(t) = R_0 + \dot{R}_0(t - t_0) + \frac{1}{2}\ddot{R}_0(t - t_0)^2 + \ldots$$

$$R(t)/R_0 = 1 + H_0(t - t_0) - \frac{q_0}{2}H_0{}^2(t - t_0)^2 + \ldots$$

For a matter dominated *Λ=0* universe the deceleration constant is another classical cosmological parameter.

$$\ddot{R} = -\frac{4\pi G}{3}R(\rho + 3p/c^2) \Rightarrow q_0 = \Omega_m/2$$

(following M. Plionis' notes or Peacock 1999)

The age of the Universe

Using $H(z) = H_0 E(z)$ at the present epoch, we have $\dot{R}/R_\circ = H_\circ E(z)/(1+z)$ and from

$$R \propto (1+z)^{-1} \Rightarrow \quad \mathrm{d}R/R_\circ = -\mathrm{d}z/(1+z)^2 \quad \Rightarrow dt = \frac{-1}{H_0}\frac{dz}{E(z)(1+z)} \tag{25}$$

we obtain the age of the Universe:

$$\boxed{t_\circ = \frac{1}{H_\circ}\int_0^\infty \frac{\mathrm{d}z}{(1+z)E(z)}} \tag{26}$$

For example, in an *Einstein-de Sitter* universe ($\Omega_\Lambda = \Omega_\mathrm{k} = 0$) we have:

$$t_\circ = \frac{2}{3H_\circ} \tag{27}$$

while for a $\Omega_\Lambda > 0$ model we obtain:

$$t_\circ^\Lambda = \frac{2}{3H_\circ}\frac{1}{\sqrt{\Omega_\Lambda}}\sinh^{-1}\left[\sqrt{\frac{\Omega_\Lambda}{\Omega_\mathrm{m}}}\right] \tag{28}$$

We therefore see that if $\Omega_\Lambda > 0$ we have that the age of the Universe is larger than what is predicted in an Einstein-de Sitter Universe.

(following M. Plionis' notes)

The age of the Universe

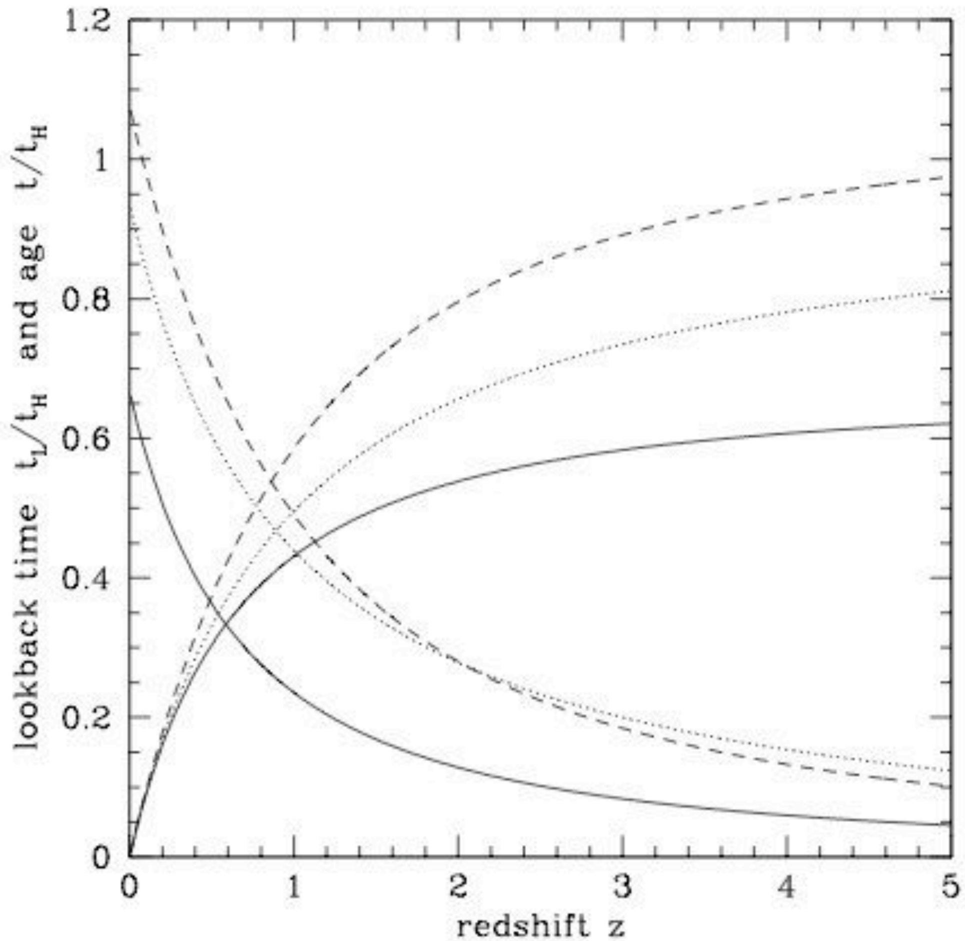
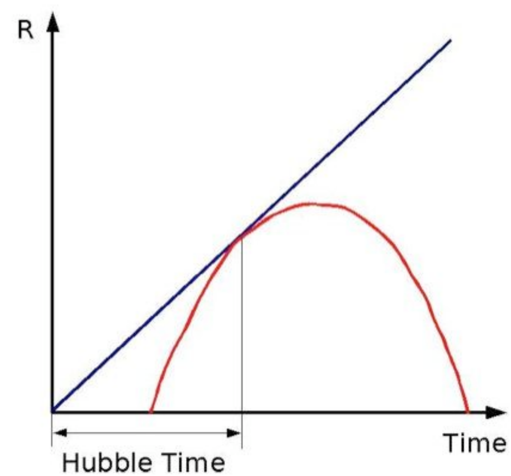


Figure 6: The dimensionless lookback time t_L/t_H and age t/t_H. Curves cross at the redshift at which the Universe is half its present age. The three curves are for the three world models, $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; $(0.05, 0)$, dotted; and $(0.2, 0.8)$, dashed.

$t_H \equiv 1/H_0$ Hubble time =

$3.09\times10^{17}\ h^{-1}$ s =
$9.80\times10^{9}\ h^{-1}$ yr



$$t_L = t_H \int \frac{dz}{(1+z)E(z)}$$

(Hogg 1999, astro-ph/9905116)