Proper distance

for a given epoch t

The *proper distance* is defined by the light-travel time along a null geodesic. If a light signal is emitted at a galaxy $\mathcal{L}_1$ at some time and received by an observer at $\mathcal{L}_0$ at another time, then these events are connected only by the light signal and since all observers must measure the same speed of light, it defines a very fundamental concept of distance.

Using the $Robertson-Walker$ metric a light signal emitted from the coordinate position (r_1, θ_0, ϕ_0) at time $t=0$ will reach (r_0, θ_0, ϕ_0) in time t determined by:

$$\int_{t_1}^{t_0} \frac{cdt}{R(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} \qquad (125)$$

where r_1 is the dimensionless comoving coordinate distance and $R(t)$ is the scale factor of the Universe.

The *proper* distance at time t is defined as:

$$d_{pro}(t) = R(t) \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = \left\{ \begin{array}{ll} R(t)\ \sin^{-1} r_1 & k=+1 \\ R(t)\ r_1 & k=0 \\ R(t)\ \sinh^{-1} r_1 & k=-1 \end{array} \right\} \qquad (126)$$

(also called transverse comoving distance, $D_M = d_{pro}(z=0)$ e.g. Hogg 1999)

and therefore at time t_0 (present epoch):

$$\boxed{D_M \approx R(t_0) r_1} \qquad (127)$$

where, with the $R \propto (1+z)^{-1}$, we have:

$$dt = \frac{-1}{H_0} \frac{dz}{E(z)(1+z)}$$

$$r_1 \simeq \int_{t_1}^{t_0} \frac{cdt}{R(t)} = \frac{1}{R(t_0)} \int_{t_1}^{t_0} c(1+z)dt = \frac{1}{R(t_0)} \int_0^z \frac{c}{H(z)} \mathrm{d}z \qquad (128)$$

(From M. Plionis' notes)

Proper distance

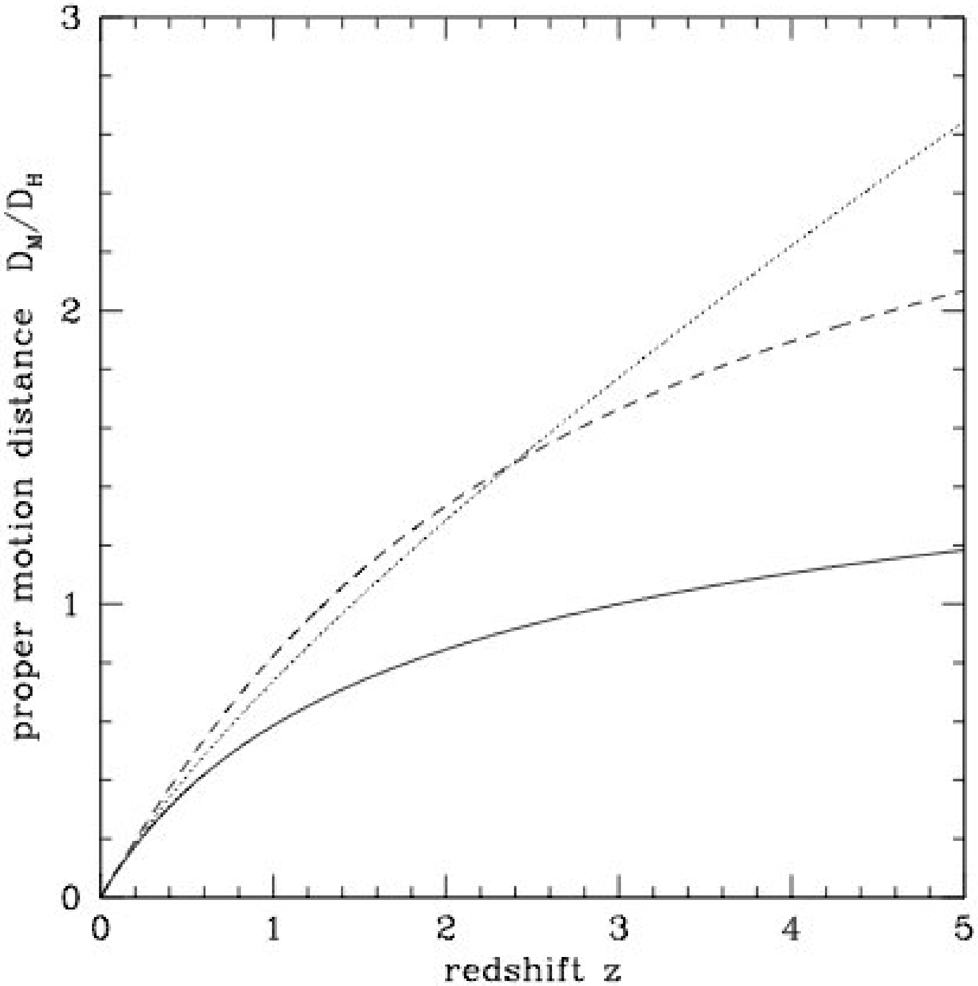


Figure 1: The dimensionless proper motion distance D_M/D_H. The three curves are for the three world models, Einstein-de Sitter $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; low-density, $(0.05, 0)$, dotted; and high lambda, $(0.2, 0.8)$, dashed.

$D_H \equiv c/H_0$ Hubble distance = 9.26×10^{25} h^{-1} m

D_M proper distance

(Hogg 1999, astro-ph/9905116)

Angular distance

The *angular diameter distance* D_A is defined as the ratio of an object's physical transverse size to its angular size (in radians). It is used to convert angular separations in telescope images into proper separations at the source. It is famous for not increasing indefinitely as $z \to \infty$; it turns over at $z \sim 1$ and thereafter more distant objects actually appear larger in angular size. Angular diameter distance is related to the transverse comoving distance by

$$D_A = \frac{D_M}{1+z} \qquad (18)$$

(Weinberg, 1972, pp 421–424; Weedman, 1986, pp 65–67; Peebles, 1993, pp 325–327). The angular diameter distance is plotted in Figure 2. At high redshift, the angular diameter distance is such that 1 arcsec is on the order of 5 8kpc (h=0.71, Ω_M=0.3, Ω_Λ=0.7)

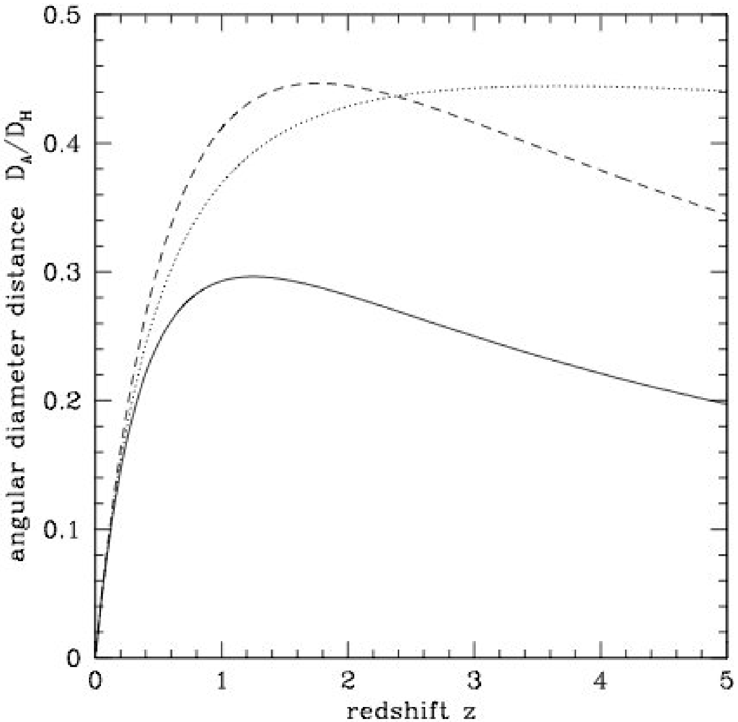


Figure 2: The dimensionless angular diameter distance D_A/D_H. The three curves are for the three world models, $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; $(0.05, 0)$, dotted; and $(0.2, 0.8)$, dashed.

The distance between (r_1, θ_1, ϕ_1) and $(r_1, \theta_1 + d\theta_1, \phi_1)$ given by the FLRW metric:

$$ds^2 = -r_1^2 R^2(t_1) d\theta_1^2 \equiv -l^2$$

$$D_A \equiv l/d\theta = r_1 R(t_1) = \frac{r_1 R(t_0)}{(1+z)} = \frac{D_M}{(1+z)}$$

(Hogg 1999, astro-ph/9905116)

Luminosity distance

If a photon is emitted at time t_1 from a source and received by us at time t_0 it will have energy at emission $E_e = \frac{hc}{\lambda_1}$ and at detection $E_d = \frac{hc}{\lambda_0} = \frac{hc}{\lambda_1(1+z)}$. If the source emits n photons isotropically in time δt_1, it has a rate of emission $n/\delta t_1$ and a rate of received photons $n/\delta t_0 = n/\delta t_1(1+z)$. The surface area at time t_0 of the sphere passing through the origin (us) and centered on the source is: $4\pi R^2(t_0)r_1^2$ (put t, r = constant in eq.(1) $\rightarrow$ line element on the surface of a Euclidean sphere of radius $r_1 R$).

Now the flux l of the source is the product of the rate of received photons, of the photon energy at reception and of the $(area)^{-1}$:

$$l = \frac{n}{\delta t_1(1+z)} \frac{hc}{\lambda_1(1+z)} \frac{1}{4\pi R^2(t_0)r_1^2} \tag{129}$$

The total luminosity (at the frequency range 1) emitted by the source is:

$$L_1 = \frac{n}{\delta t_1} \frac{hc}{\lambda_1} \tag{130}$$

From eq.(119), (129) and (130) we have that the distance r, called the *Luminosity distance* is:

$$l = \frac{L}{4\pi r^2}$$

$$\boxed{d_L = r_1 R(t_0)(1+z) = (1+z)\int_0^z \frac{c}{H(z)} \mathrm{d}z} \tag{131}$$

$$d_L = (1+z) D_M$$

(M. Plionis' notes)

Luminosity distance

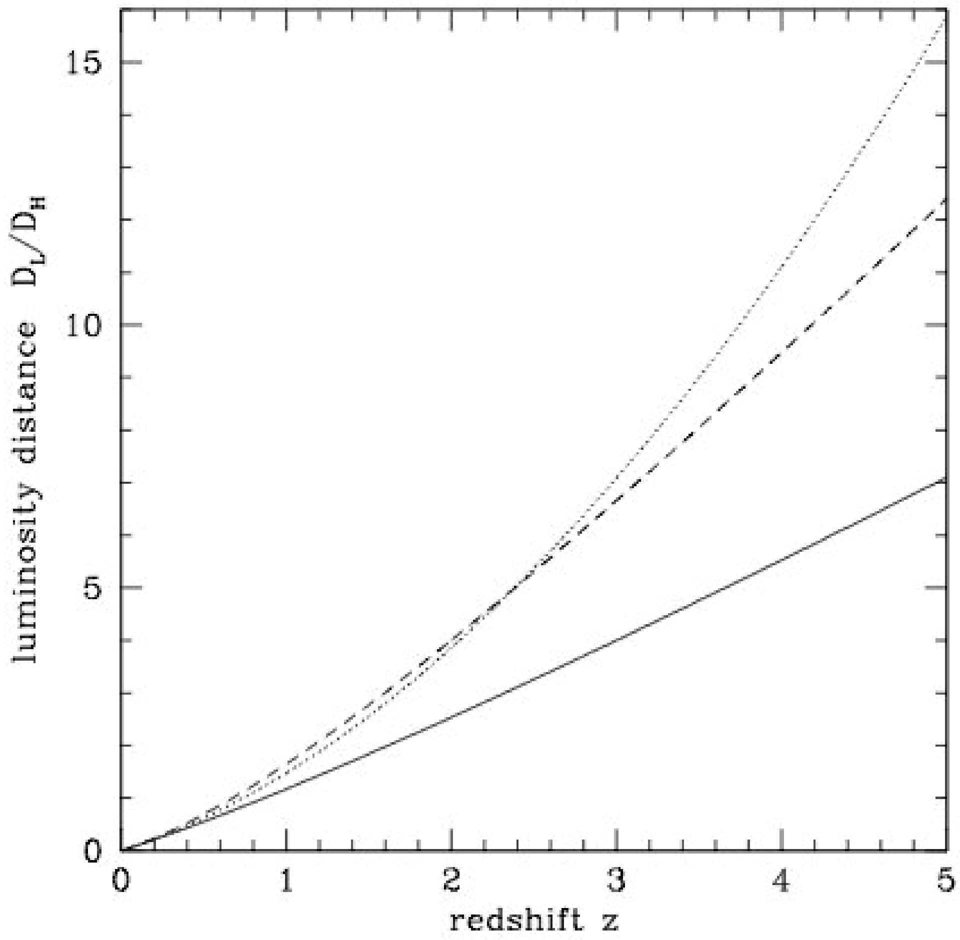


Figure 3: The dimensionless luminosity distance D_L/D_H. The three curves are for the three world models, $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; $(0.05, 0)$, dotted; and $(0.2, 0.8)$, dashed.

(Hogg 1999, astro-ph/9905116)

Luminosity distance

In order to express d_L as a function of z we need to remove the explicit dependance of d_L on the coordinate distance r_1. This is usually done by solving the Einstein field equations (see section 1.6). However, there is an easier way to obtain the value of r_1 without solving directly Einstein's field equations. This is by using a Taylor expansion of $R(t)$ around t_0:

$$R(t) = R(t_0) + (t - t_0)\left[\frac{dR}{dt}\right]_0 + \frac{(t-t_0)^2}{2}\left[\frac{d^2R}{dt^2}\right]_0 + \ldots \tag{133}$$

$\Longrightarrow$

$$\frac{R(t)}{R(t_0)} = \frac{1}{1+z} = 1 - (t_0 - t)H_\circ - \frac{1}{2}(t-t_0)^2 q_\circ H_\circ^2 + \ldots \tag{134}$$

where $q_\circ$ is the *deceleration* parameter, which defines the different cosmological models (see section 2.3). Now inverting eq.(134) we obtain:

$$t_0 - t = \frac{z}{H_\circ}\left[1 - (1 + q_\circ/2)z + \ldots\right] \tag{135}$$

and introducing this into eq.(128) we get

$$r_1 = \frac{c}{R_(t_0)H_\circ}\left[z - \frac{1}{2}(1+q_\circ)z^2 + \ldots\right] \tag{136}$$

and from eq.(131) we get:

$$\boxed{d_L = \frac{c}{H_\circ}\left[z + \frac{1}{2}(1-q_\circ)z^2 + \ldots\right]} \tag{137}$$

A more general expression, derived using Einsteins equations and valid for all cosmological models, is given by:

$$\boxed{d_L = \frac{c}{H_\circ q_o^2}\left[q_o z + (q_o - 1)\left(\sqrt{1+2zq_o} - 1\right)\right]} \tag{138}$$

(M. Plionis' notes)

Comoving volume

The *comoving volume* V_C is the volume measure in which number densities of non-evolving objects locked into Hubble flow are constant with redshift. It is the proper volume times three factors of the relative scale factor now to then, or $(1+z)^3$. Since the derivative of comoving distance with redshift is $1/E(z)$ defined in (14), the angular diameter distance converts a solid angle $d\Omega$ into a proper area, and two factors of $(1+z)$ convert a proper area into a comoving area, the comoving volume element in solid angle $d\Omega$ and redshift interval dz is

$$dV_C = D_H \frac{(1+z)^2 D_A^2}{E(z)} d\Omega \, dz \tag{28}$$

where D_A is the angular diameter distance at redshift z and $E(z)$ is defined in (14) (Weinberg, 1972, p. 486; Peebles, 1993, pp 331–333). The comoving volume element is plotted in Figure 5. The integral of the comoving volume element from the present to redshift z gives

the total comoving volume, all-sky, out to redshift z

$$V_C = \begin{cases} \left(\frac{4\pi D_H^3}{2\Omega_k}\right)\left[\frac{D_M}{D_H}\sqrt{1+\Omega_k \frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_k|}} \operatorname{arcsinh}\left(\sqrt{|\Omega_k|}\frac{D_M}{D_H}\right)\right] & \text{for } \Omega_k > 0 \\ \frac{4\pi}{3} D_M^3 & \text{for } \Omega_k = 0 \\ \left(\frac{4\pi D_H^3}{2\Omega_k}\right)\left[\frac{D_M}{D_H}\sqrt{1+\Omega_k \frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_k|}} \arcsin\left(\sqrt{|\Omega_k|}\frac{D_M}{D_H}\right)\right] & \text{for } \Omega_k < 0 \end{cases} \tag{29}$$

(Carrol, Press & Turner, 1992), where D_H^3 is sometimes called the *Hubble volume*. The comoving volume element and its integral are both used frequently in predicting number counts or luminosity densities.

(Hogg 1999, astro-ph/9905116)

Comoving volume

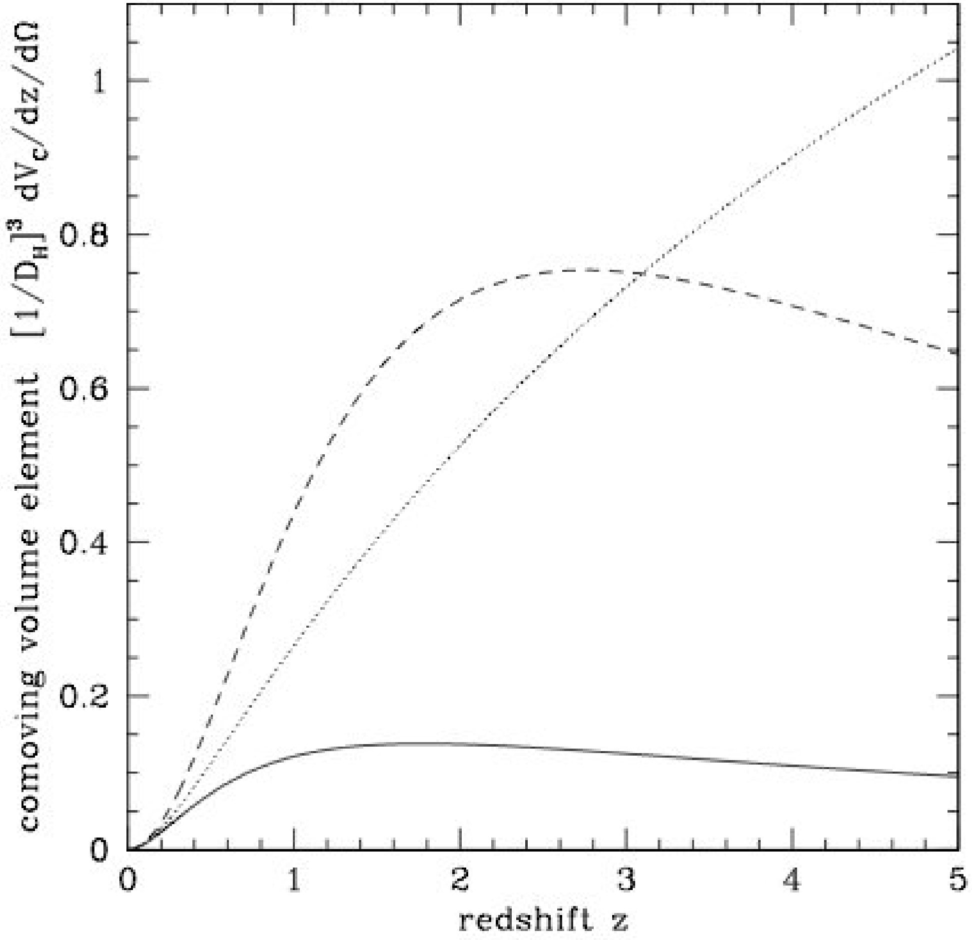


Figure 5: The dimensionless comoving volume element $(1/D_{\rm H})^3\,(dV_{\rm C}/dz)$. The three curves are for the three world models, $(\Omega_{\rm M}, \Omega_\Lambda) = (1,0)$, solid; $(0.05,0)$, dotted; and $(0.2,0.8)$, dashed.

(Hogg 1999, astro-ph/9905116)

Models with Λ=0

⊕ **$k = 0, \Omega = 1$ - FLAT:** THE UNIVERSE EXPANDS FOREVER BUT WITH A DECREASING RATE.

It is straightforward to solve eq.(12) with $k = 0$, to obtain

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 + \frac{\Lambda c^2}{3}R^2 - kc^2$$

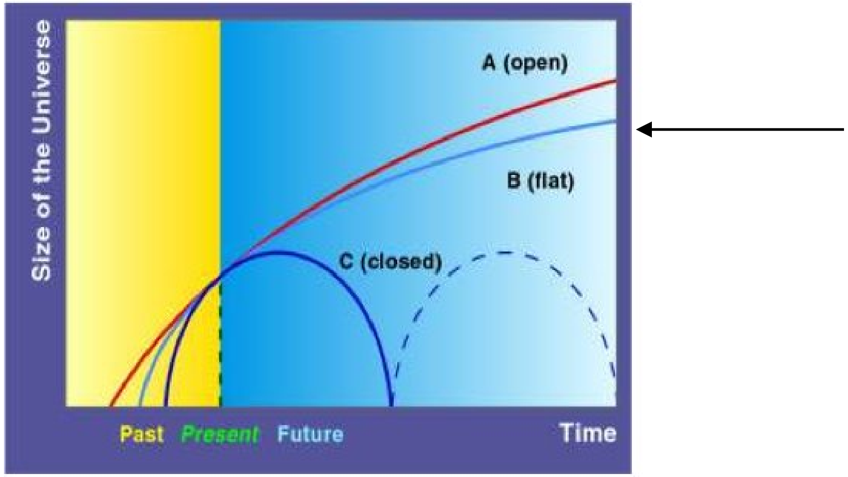
$$\boxed{R(t) \propto t^{\frac{2}{3(1+w)}}} \quad (34)$$

and using eq.(16) and eq.(12) we obtain the behaviour of the density as a function of cosmic time:

$$\rho \propto R^{-3(1+w)}$$

$$\boxed{\rho = \frac{1}{6\pi G t^2 (1+w)^2}} \quad (35)$$

and finally differentiating eq.(34) we can get the age of the Universe $t_0 = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)E(z)} = \frac{2}{3(1+w)H_0}$



(M. Plionis' notes)

Einstein-de Sitter Universe

This is a universe with $\Omega_m = 1$, $\Omega_\Lambda = 0$, i.e. the universe is Euclidean:

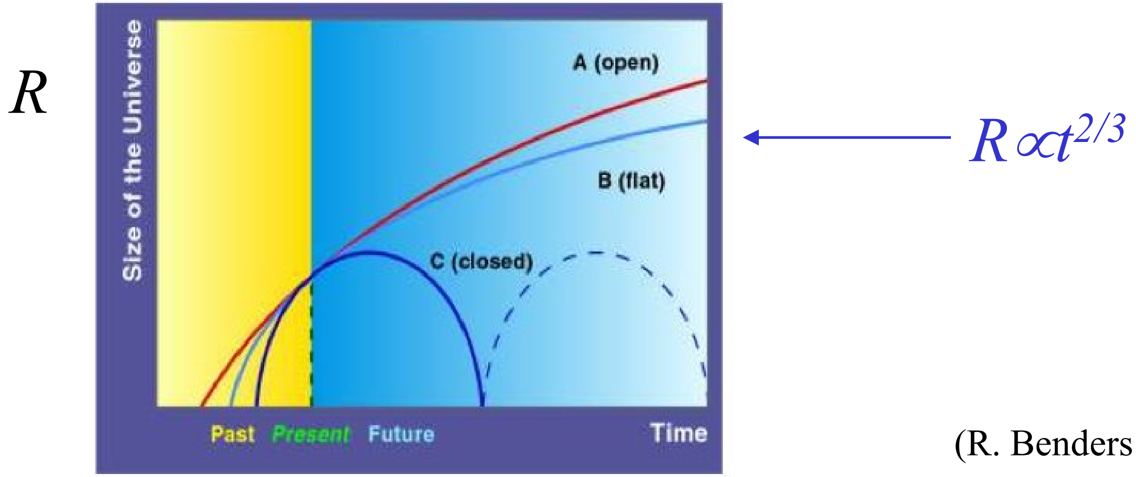
$$\dot{R}^2 = \frac{8\pi G\varrho}{3} R^2$$

which can be integrated and yields:

$$R^{1/2}\, dR = \left(\frac{8\pi G\varrho_0}{3}\right)^{1/2} dt$$

Using the definition of Ω_m (13.8) and considering that we assumed $\Omega_m = 1$, we have $H_0^2 = (8\pi G\varrho_0)/3$ and thus:

$$\boxed{R = \left(\frac{3}{2} H_0\, t\right)^{2/3} \qquad (p = 0, \Lambda = 0, \Omega_m = 1)}$$



(R. Benders' notes)

Models with Λ=0

- **k = +1, Ω > 1 - CLOSED**: THE UNIVERSE EXPANDS UNTIL $(8\pi G\rho R^2)/3 = c^2$ (AT z_c), THEREAFTER IT STARTS CONTRACTING.

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 + \frac{\Lambda c^2}{3}R^2 - kc^2$$

From eq.(11) we have:

$$\varepsilon \equiv kc^2/2$$

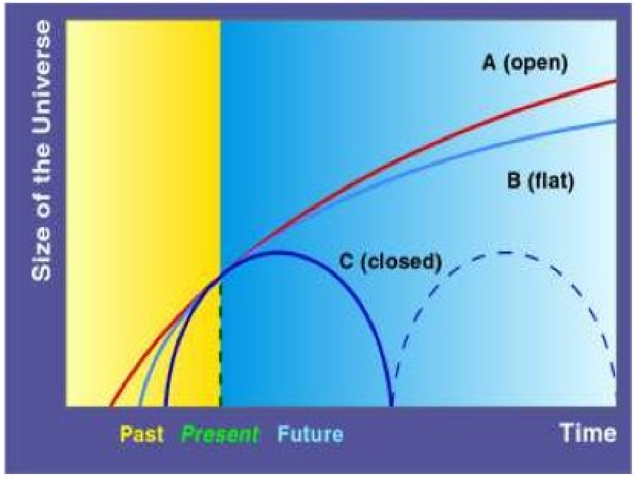
$$\dot{R} = \left[\frac{2(GM - |\mathcal{E}|R)}{R}\right]^{\frac{1}{2}} \quad (36)$$

Using

$$R = \frac{GM}{2|\mathcal{E}|}(1 - \cos\psi) = \frac{GM}{|\mathcal{E}|}\sin^2\frac{\psi}{2} \quad (37)$$

differentiating the above equation, inserting it in eq.(36) and then integrating, we obtain after some algebra:

$$t = \frac{GM}{(2|\mathcal{E}|)^{3/2}}(\psi - \sin\psi) \quad (38)$$



(M. Plionis' notes)

Models with Λ=0

It is evident for this model that at some time, t_m, the universe reaches its maximum expansion scale, $R_{\max}$, after which it starts recontracting. This time corresponds to $\psi = \pi$ as can be easily seen from the above relations. We can therefore parametrize the solutions as:

$$\frac{R}{R_{\max}} = \frac{1}{2}(1 - \cos\psi) \tag{39}$$

$$\frac{t}{t_{\max}} = \frac{1}{\pi}(\psi - \sin\psi) \tag{40}$$

Using the definitions (17) and (18) at the present time and eq.(24) we obtain:

$$R_\circ = \frac{(2|\mathcal{E}|)^{1/2}}{H_\circ} \frac{1}{(\Omega_\circ - 1)^{1/2}}$$

and putting this into eq.(36) we have

$$GM = \frac{\Omega_\circ (2|\mathcal{E}|)^{3/2}}{2H_\circ} \frac{1}{(\Omega_\circ - 1)^{3/2}} \tag{41}$$

In order therefore to have $R = R_\circ$ we must have

$$\frac{(2|\mathcal{E}|)^{1/2}\Omega_\circ}{4H_\circ} \frac{1 - \cos\psi_\circ}{(\Omega_\circ - 1)^{3/2}} = \frac{(2|\mathcal{E}|)^{1/2}}{H_\circ} \frac{1}{(\Omega_\circ - 1)^{1/2}}$$

which then gives the present value of ψ, which is:

$$\psi_\circ = \cos^{-1}\left[\frac{2 - \Omega_\circ}{\Omega_\circ}\right]$$

and therefore substituting this and eq.(41) into eq.(38) we can easily obtain the present age of the universe.

(M. Plionis' notes)

Models with Λ=0

- **k = −1, Ω < 1 - OPEN:** UNIVERSE EXPANDS FOREVER SINCE $\dot{R}^2 > 0 \ \forall\, t$.

  In this case we can rewrite eq.(10) as:

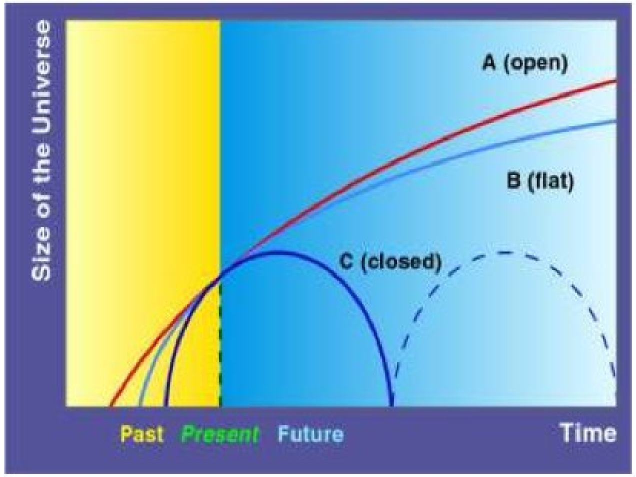
$$\dot{R} = \left[\frac{GM + |\mathcal{E}|R}{R}\right]^{\frac{1}{2}} \tag{42}$$

and, similarly with the $k = 1$ case, we find the parametric solutions:

$$\boxed{R = \frac{GM}{2|\mathcal{E}|}(\cosh\phi - 1) \qquad t = \frac{GM}{(2|\mathcal{E}|)^{3/2}}(\sinh\phi - \phi)} \tag{43}$$

from which we find that the present day value of ϕ is:

$$\phi_\circ = \cosh^{-1}\left[\frac{2 - \Omega_\circ}{\Omega_\circ}\right]$$



(M. Plionis' notes)

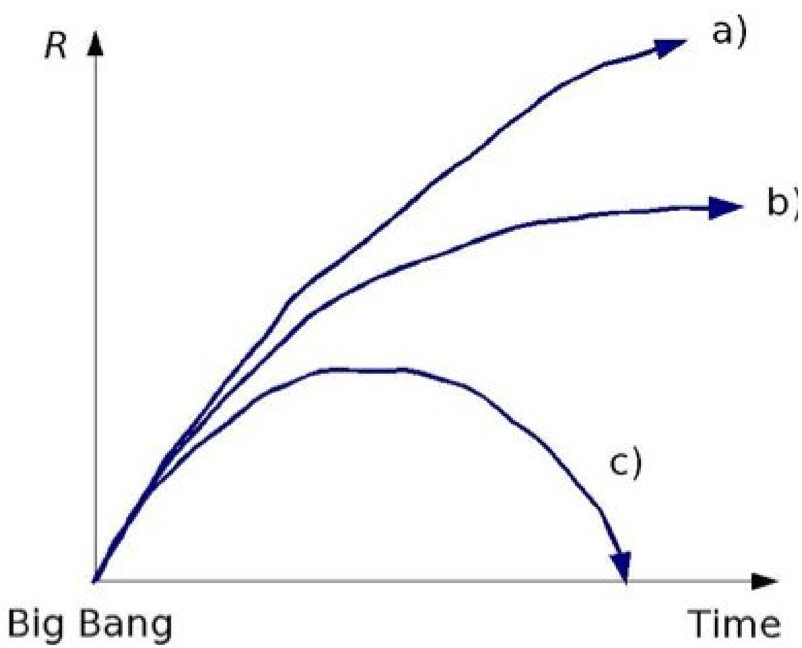
Models with Λ=0

Summary

a) $k = -1$ $q < ½$ $\Omega < 1$
open, hyperbolic Universe

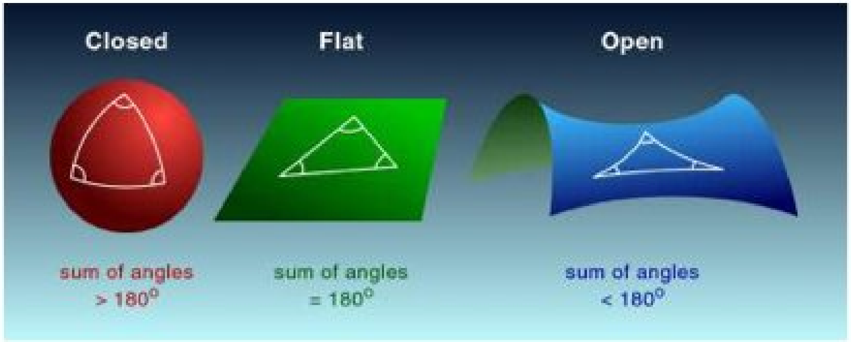
b) $k = 0$ $q = ½$ $\Omega = 1$
open, flat Universe

c) $k = 1$ $q > ½$ $\Omega > 1$
closed, spherical Universe



Curvature of the space-time continuum k:

k > 0	**k = 0**	**k < 0**
positive curvature	zero curvature	negative curvature



The role of Λ and k

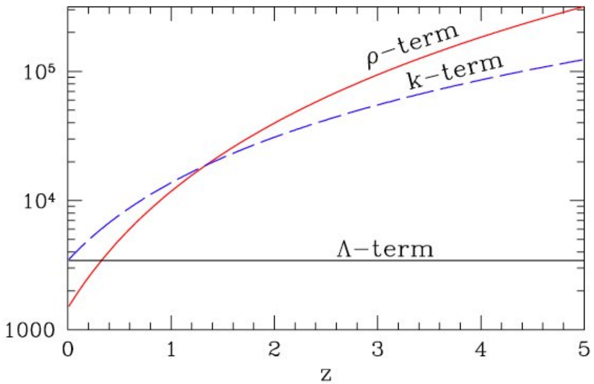
Due to the recent interest in the $\Lambda > 0$, $k = 0$ Universes, it is important to investigate the dynamical effects that this term may have in the evolution of the Universe and thus also in the structure formation processes (see Fig.2). We realize these effects by inspecting the magnitudes of the two terms in the right hand side of (12). We have the density term:

$$H = H_0\left[\Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda\right]^{1/2} \tag{29}$$

By equating the above two terms we can find the redshift at which they have equal contributions to the dynamics of the Universe. Evidently this happens only in the very recent past:

$$z_c = \left(\frac{\Omega_\Lambda}{\Omega_m}\right)^{1/3} - 1 \tag{30}$$

Observations suggest that $\Omega_m \simeq 0.3$ and $\Omega_\Lambda \simeq 0.7$, and therefore we have $z_c \simeq 0.3$, which implies that the present dynamics of the universe are dominated by the Λ-term, although for the largest part of the history of the Universe the determining factor in shaping its dynamical evolution is the matter content.



(M. Plionis' notes)

Models with $\Lambda \neq 0$

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 + \frac{\Lambda c^2}{3}R^2 - kc^2$$

From (12) we see that if $k \leq 0$, then $\dot{R}^2$ is always nonnegative for $\Lambda > 0$, and thus the universe expands for ever, while if $\Lambda < 0$ then the universe can expand and then recontract again (as in the $k = 1$, $\Lambda = 0$ case).

The recent SNIa observations (see section 3.2) and the CMB power-spectrum results (see section 3.1) have shown that the *Standard* Cosmological paradigm should be considered that of a flat, $\Omega_\Lambda \simeq 0.7$, $\Omega_{\rm m} \simeq 0.3$ model. Thus we will consider such a model in the following discussion. Evaluating (12) at the present epoch, changing successively variables: $x = R^{3/2}$, $y = x(\Omega_{\rm m}/\Omega_\Lambda)^{1/2}\ R_\circ^{-3/2}$ and $\theta = \sinh^{-1} y$ and then integrating, we obtain:

$$\boxed{t = \frac{2}{3H_\circ\sqrt{\Omega_\Lambda}} \sinh^{-1}\left[\left(\frac{\Omega_\Lambda}{\Omega_{\rm m}}\right)^{\frac{1}{2}} \left(\frac{R}{R_\circ}\right)^{\frac{3}{2}}\right]} \tag{46}$$

and

$$R = R_\circ \left(\frac{\Omega_{\rm m}}{\Omega_\Lambda}\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}}\left(\frac{3H_\circ\sqrt{\Omega_\Lambda}}{2}\,t\right) \tag{47}$$

It is interesting to note that in this model there is an epoch which corresponds to a value of $R = R_{\rm I}$, where the expansion slows down and remains in a quasi-stationary phase for some time, expanding with $\ddot{R} > 0$ thereafter (see Fig.2). At the quasi-stationary epoch, called the inflection point, we have $\ddot{R} = 0$ and thus from (12) by differentiation we have:

$$R_{\rm I} = \left(\frac{\Omega_m}{2\,\Omega_\Lambda}\right)^{\frac{1}{3}} R_\circ \tag{48}$$

(M. Plionis' notes)

Models with Λ≠0

Now from (46) and (48) we have that the age of the universe at the inflection point is:

$$t_{\rm I} = \frac{2}{3H_\circ\sqrt{\Omega_\Lambda}} \sinh^{-1}\left(\sqrt{\frac{1}{2}}\right) . \qquad (49)$$

The Hubble function at $t_{\rm I}$ is:

$$H(t_{\rm I}) = H_\circ\sqrt{\Omega_\Lambda}\coth\left(\frac{3H_\circ\sqrt{\Omega_\Lambda}}{2}\, t_{\rm I}\right) \Longrightarrow H_{\rm I} = H_\circ\sqrt{3\Omega_\Lambda}$$

so if $t_\circ > t_{\rm I}$ we must have $H_\circ < H_{\rm I}$.

This is an important result because it indicates that introducing an Ω_Λ-term, and if we live at a time that fulfils the condition $t_\circ > t_{\rm I}$, we can increase the age of the universe to comfortably fit the globular cluster ages while keeping the value of $\Omega_{\rm m} < 1$ and also a flat ($\Omega_{\rm k} = 0$) space geometry. From (49) and (28) and for the preferred values $\Omega_\Lambda = 0.7$ and $\Omega_{\rm m} = 0.3$ we indeed obtain $t_\circ/t_{\rm I} \simeq 1.84$ (see also Fig.2), which implies that we live in the accelerated phase of the Universe. Note that in order for the present time ($t_\circ$) to be in the accelerated phase of the expansion we must have: $\Omega_\Lambda > 1/3$.

(M. Plionis' notes)

Models with Λ≠0

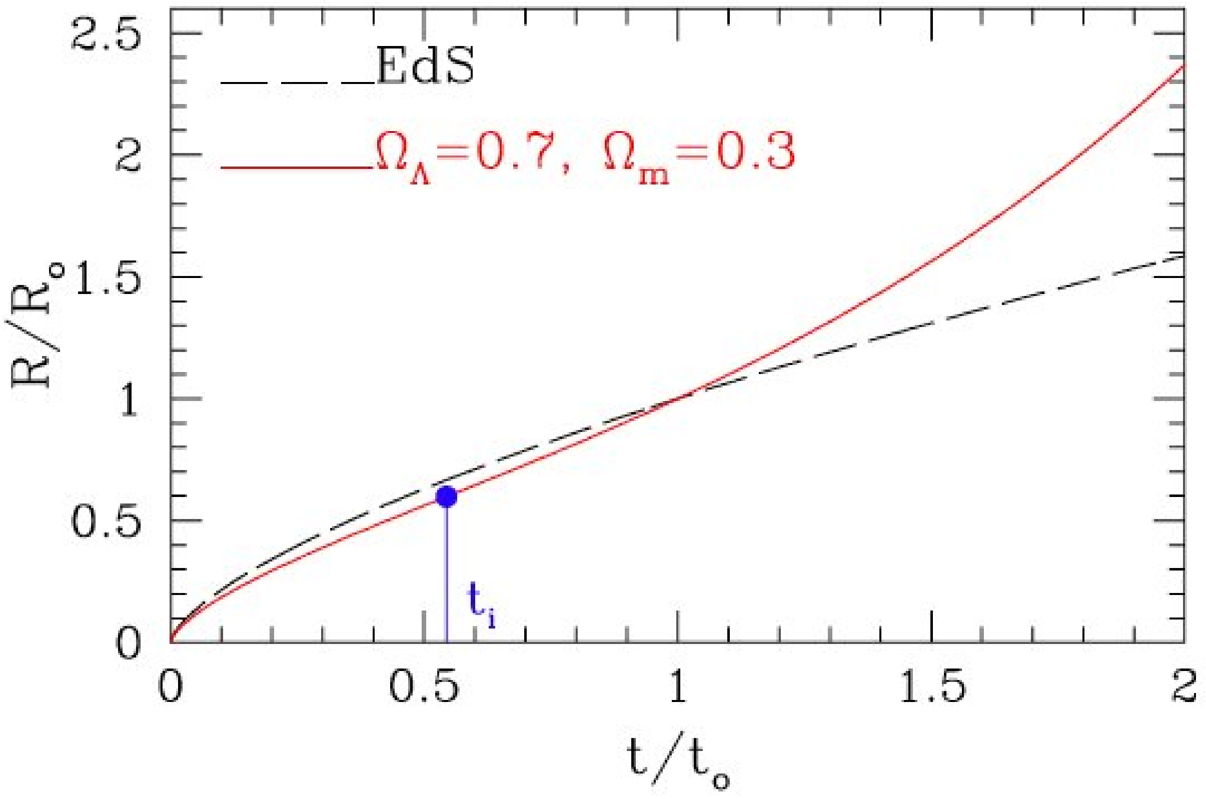


Figure 2: The expansion of the Universe in an Einstein de-Sitter (EdS) and in the *preferred* Λ model. We indicate the inflection point beyond which the expansion accelerates. It is evident that in this model we live in the accelerated regime and thus the age of the Universe is larger than the Hubble time ($H_\circ^{-1}$).